Allowable Complex Metrics in Quantum Cosmology

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Introduction: gravitational path integrals

$$\Psi = \int \mathcal{D}\boldsymbol{g}\, e^{\frac{i}{\hbar}S[\boldsymbol{g}]} = \int \mathcal{D}\boldsymbol{g}\, \exp\left(\frac{i}{16\pi G_{\mathrm{N}}\hbar}\int \mathrm{d}^4x\,\sqrt{-g}\,R\right)$$

- ‘Simple’ way of combining general relativity and quantum mechanics

- Should provide us with insights about quantum gravity
(at least at the semi-classical level)

- How should the path integrals be defined?
Which geometries should be included in the sum?

- This is a quantum problem where the metric is generally complex
(in particular, Lorentzian, Euclidean, or anything in between)

Example of metric that should be non-allowable

Witten [arXiv:2111.06514]

- Flat spacetime solution to vacuum Einstein equations:

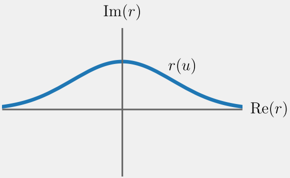
$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \qquad r > 0$$

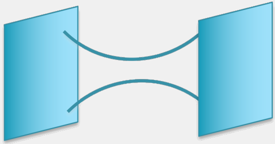
$\longrightarrow$ Minkowski, zero-action solution

- Promote r to a function in the complex plane, with real parameter u:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + r'(u)^2\mathrm{d}u^2 + r(u)^2\mathrm{d}\Omega^2$$

$\longrightarrow$ different (complex) geometry, e.g., wormhole, but still flat and so still a zero-action solution, therefore same likelihood as Minkowski





$\Rightarrow$ such complex solutions must presumably be eliminated from the path integral, i.e., they should be non-allowable

A proposal: the Kontsevich-Segal criterion

Kontsevich & Segal [arXiv:2105.10161], Witten [2111.06514]

- A metric is said to be allowable if a generic QFT can be defined on it, in the sense that its path integral converges for the actions of all real scalar and gauge fields (since they have local covariant energy-momentum tensor)
- E.g., E&M with $S[\boldsymbol{A}] = -\frac{1}{4}\int \mathrm{d}^4x\,\sqrt{-g}\,g^{ac}g^{bd}F_{ab}F_{cd}\,,\ F_{ab} = \partial_a A_b - \partial_b A_a$

$$\int \mathcal{D}\boldsymbol{A}\, e^{\frac{i}{\hbar}S[\boldsymbol{A}]}\ \text{converges} \quad \Leftrightarrow \quad \left| e^{\frac{i}{\hbar}S[\boldsymbol{A}]}\right| < 1 \quad \Leftrightarrow \quad \mathrm{Re}\left(\sqrt{g}\,g^{ac}g^{bd}F_{ab}F_{cd}\right) > 0$$

- Another example, real massive scalar field with $S[\phi] = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(\partial^a\phi\partial_a\phi + m^2\phi^2\right)$

$$\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}S[\phi]}\ \text{converges} \quad \Leftrightarrow \quad \mathrm{Re}\left(\sqrt{g}\right) > 0\,, \quad \mathrm{Re}\left(\sqrt{g}\,g^{ab}\partial_a\phi\partial_b\phi\right) > 0$$

- For a generic p-form gauge field $\boldsymbol{A}_p$ with field strength $\boldsymbol{F}_{p+1} = \mathbf{d}\boldsymbol{A}_p$:

$$\mathrm{Re}\left(\sqrt{g}\,g^{a_1b_1}\cdots g^{a_{p+1}b_{p+1}}F_{a_1\cdots a_{p+1}}F_{b_1\cdots b_{p+1}}\right) > 0$$

Kontsevich-Segal: a more convenient formulation

Kontsevich & Segal [2105.10161], Witten [2111.06514], Lehners [2111.07816]

- Let us diagonalize the metric:

$$g_{ab} = \lambda_{(a)}\delta_{ab} \quad \Rightarrow \quad \sqrt{g} = \prod_a \sqrt{\lambda_{(a)}}$$

- The Kontsevich-Segal criterion becomes

$$\mathrm{Re}\left(\sqrt{g}\prod_a \lambda_{(a)}^{-1}\right) > 0 \quad \Leftrightarrow \quad \Sigma \equiv \sum_a \left|\mathrm{Arg}\left(\lambda_{(a)}\right)\right| < \pi$$

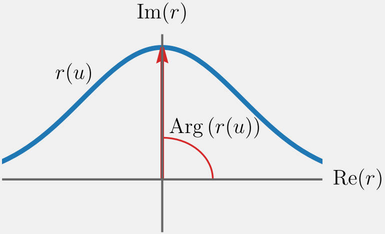
Let's revisit the complex wormhole

Witten [2111.06514]

$$\mathrm{d}s^2 \supset r(u)^2 \mathrm{d}\Omega_{(3)}^2 \quad \Rightarrow \quad \Sigma(u) \geq 3 \left| \mathrm{Arg}\left(r(u)^2\right) \right|$$

- When $r(u)$ crosses the imaginary axis:

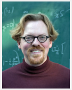
$$\mathrm{Arg}\left(r(u)\right) = \frac{\pi}{2} \quad \Rightarrow \quad \Sigma(u) \geq 3\pi$$



$\Rightarrow$ Impossible to satisfy the Kontsevich-Segal bound $\Sigma < \pi$ everywhere, therefore this complex wormhole metric should not be allowed in a path integral

$\Rightarrow$ Resolves the "same-likelihood-as-Minkowski" paradox

Let's apply this to (quantum) cosmology!

Caroline Jonas, Jean-Luc Lehners & JQ [soon to appear on arXiv]

- Let's check the bound for a real Lorentzian FLRW metric:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + a(t)^2\mathrm{d}\mathbf{x}^2 \quad \Rightarrow \quad \Sigma(t) = \left|\mathrm{Arg}\left(-N^2\right)\right| + 3\left|\mathrm{Arg}\left(a(t)^2\right)\right| = \pi$$

$\Rightarrow$ Lorentzian metrics are on the boundary of the Kontsevich-Segal bound

$\Rightarrow$ Conditionally convergent path integral

$\rightarrow$ Lorentzian metrics can be regulated as in QFT with Feynman's $i\epsilon$ prescription (about that, see also Visser [2111.14016]), e.g.:

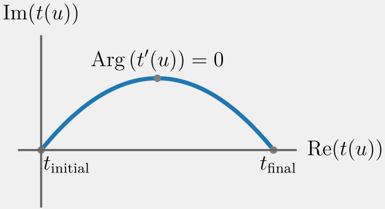
$$\mathrm{d}s^2 = -(1 \mp i\epsilon)\mathrm{d}t^2 + a(t)^2\mathrm{d}\mathbf{x}^2$$

$\rightarrow$ Could we also have a complex time path?

$$\mathrm{d}s^2 = -t'(u)^2\mathrm{d}u^2 + a(t(u))^2\mathrm{d}\mathbf{x}^2 \Rightarrow \Sigma(u) = \left|\mathrm{Arg}\left(-t'(u)^2\right)\right| + 3\left|\mathrm{Arg}\left(a(t(u))^2\right)\right|$$

→ Could we also have a complex time path? No!

$$\Sigma(u) = \left|\mathrm{Arg}\left(-t'(u)^2\right)\right| + 3\left|\mathrm{Arg}\left(a(t(u))^2\right)\right|$$



⇒ For the time path to start and end on the real line, it must ‘turn around’, and at that point $\Sigma(u) \geq \pi$

de Sitter 'classical' transitions

- Consider classical transitions between scale factor values $\sqrt{q_0}$ and $\sqrt{q_1}$ for GR with a cosmological constant in closed FLRW ($8\pi G_{\rm N} = 1$):

$$\Psi = \int_{\mathcal{C}} \mathcal{D}N \int_{q_0}^{q_1} \mathcal{D}q \, e^{\frac{i}{2\hbar} \int \mathrm{d}^4x \sqrt{-g}(R-2\Lambda)} \,, \quad \mathrm{d}s^2 = -\frac{N^2}{q(t)}\mathrm{d}t^2 + q(t)\mathrm{d}\Omega_{(3)}^2$$

$\rightarrow$ With this metric ansatz, action becomes quadratic, hence solvable
Interpretation: $q(t) = a(t)^2$, $N\mathrm{d}t/\sqrt{q(t)} = \mathrm{d}t_{\rm physical}$

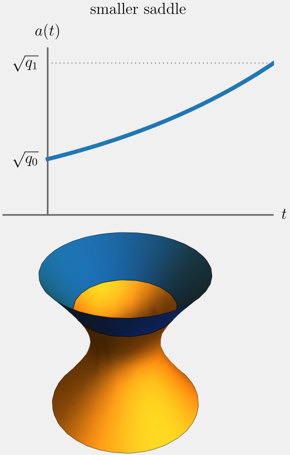
$\rightarrow$ We are left with an ordinary integral over the lapse function, which has 4 saddle points:

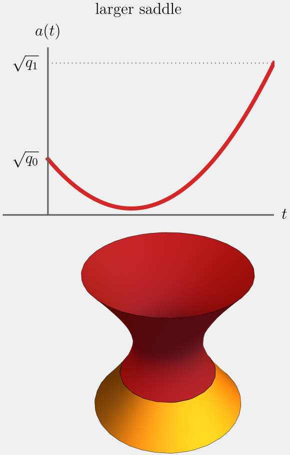
$$\Psi = \int \frac{\mathrm{d}N}{\sqrt{N}} e^{\frac{i}{\hbar}\left(N^3\frac{\Lambda^2}{36} + N\left(3 - \frac{\Lambda}{2}(q_1+q_0)\right) - \frac{3}{4N}(q_1-q_0)^2\right)}$$

$$N_{\rm saddles} = \frac{3}{\Lambda}\left(\pm\sqrt{\frac{\Lambda}{3}q_1 - 1} \pm \sqrt{\frac{\Lambda}{3}q_0 - 1}\right)$$

de Sitter 'classical' transitions

- The 4 saddle points come in pairs (time reversal of each other)
- The smaller ones describe de Sitter expansion;
  the larger ones a de Sitter bounce

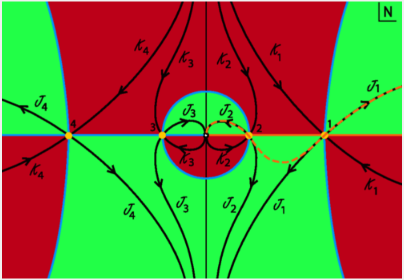




de Sitter ‘classical’ transitions

- Find appropriate complex lapse integration contour $\longrightarrow$ difficult problem

  Feldbrugge, Lehners, Turok [1703.02076]

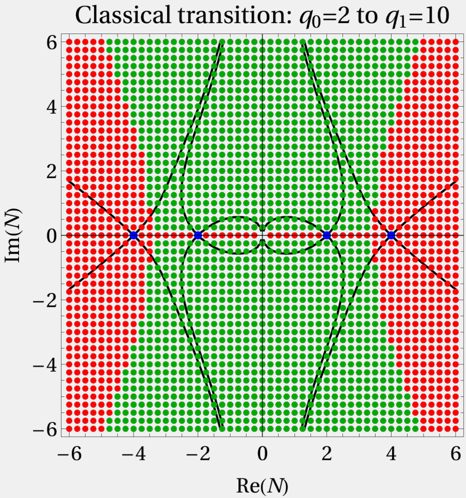


- One usually obtains a superposition of expanding and bouncing saddles
- But are all the geometries in the path allowable?

$\rightarrow$ We propose to apply the Kontsevich-Segal criterion:

$$\Sigma(u) = \left|\mathrm{Arg}\left(-\frac{N^2}{q(t(u))}t'(u)^2\right)\right| + 3\left|\mathrm{Arg}\left(q(t(u))\right)\right| < \pi \quad \forall\, t(u)$$

What we find



Blue dots: saddle point geometries
Green dots: allowable geometries
Red dots: non-allowable geometries
Black lines: steepest descent contours

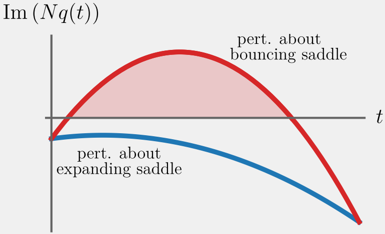
⇒ Steepest descent contours are cut off on real-N line

⇒ Bouncing saddles are unreachable!

Illustration

- What goes wrong for the geometries that exclude the bouncing saddles?
- Consider, e.g., a path integral that contains a real scalar field mass term:

$$\int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\int \mathrm{d}t \, Nq(t)\left(-\frac{1}{2}m^2\phi^2\right)} \text{ converges} \quad \Leftrightarrow \quad \mathrm{Im}\,(Nq(t)) < 0$$



- When $\mathrm{Im}\,(Nq(t)) > 0$, the integration over ϕ is divergent for an infinite range of field values $\Rightarrow$ ill-defined path integral

Discussion and conclusions

- A restriction on complex metrics is required
- The proposal of Kontsevich-Segal is promising, as it eliminates pathological metrics, while retaining sensible ones.
  In quantum cosmology:
  - $\rightarrow$ steepest descent contours are cut off
  - $\rightarrow$ bounces tend to be unreachable
- I could have presented many other results, in particular for quantum transitions (such as the no-boundary proposal), so stay tuned for the paper!

Some caveats:

- Swampland-like conjecture might force fields to have restricted ranges, in which case divergences might not occur
- Restricted to minisuperspace. More general geometries may be allowable, thus potentially changing some of the conclusions

Thank you for your attention!

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