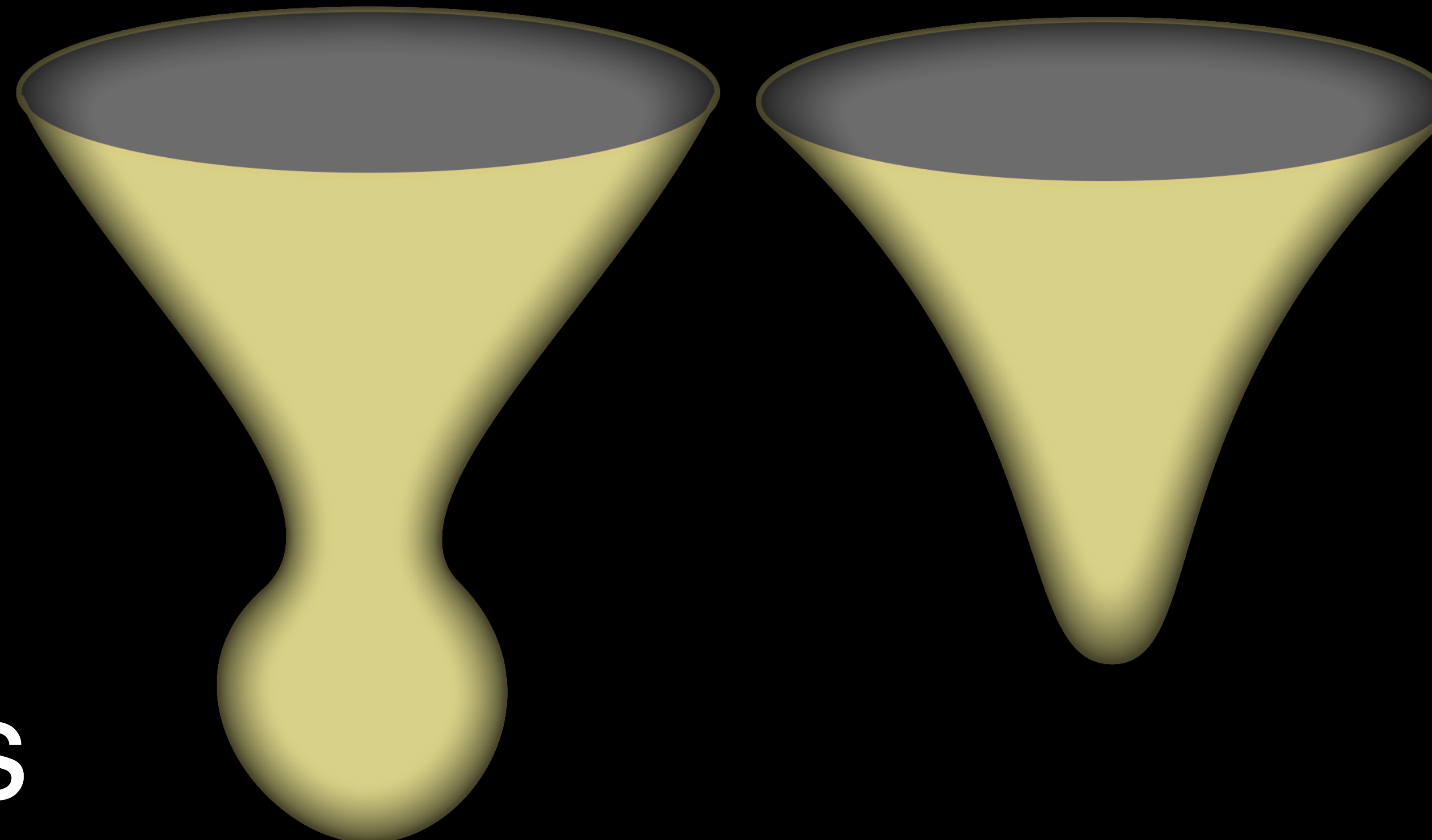
A Tale of Complex Metrics in Gravitational Path Integrals



Jerome Quintin

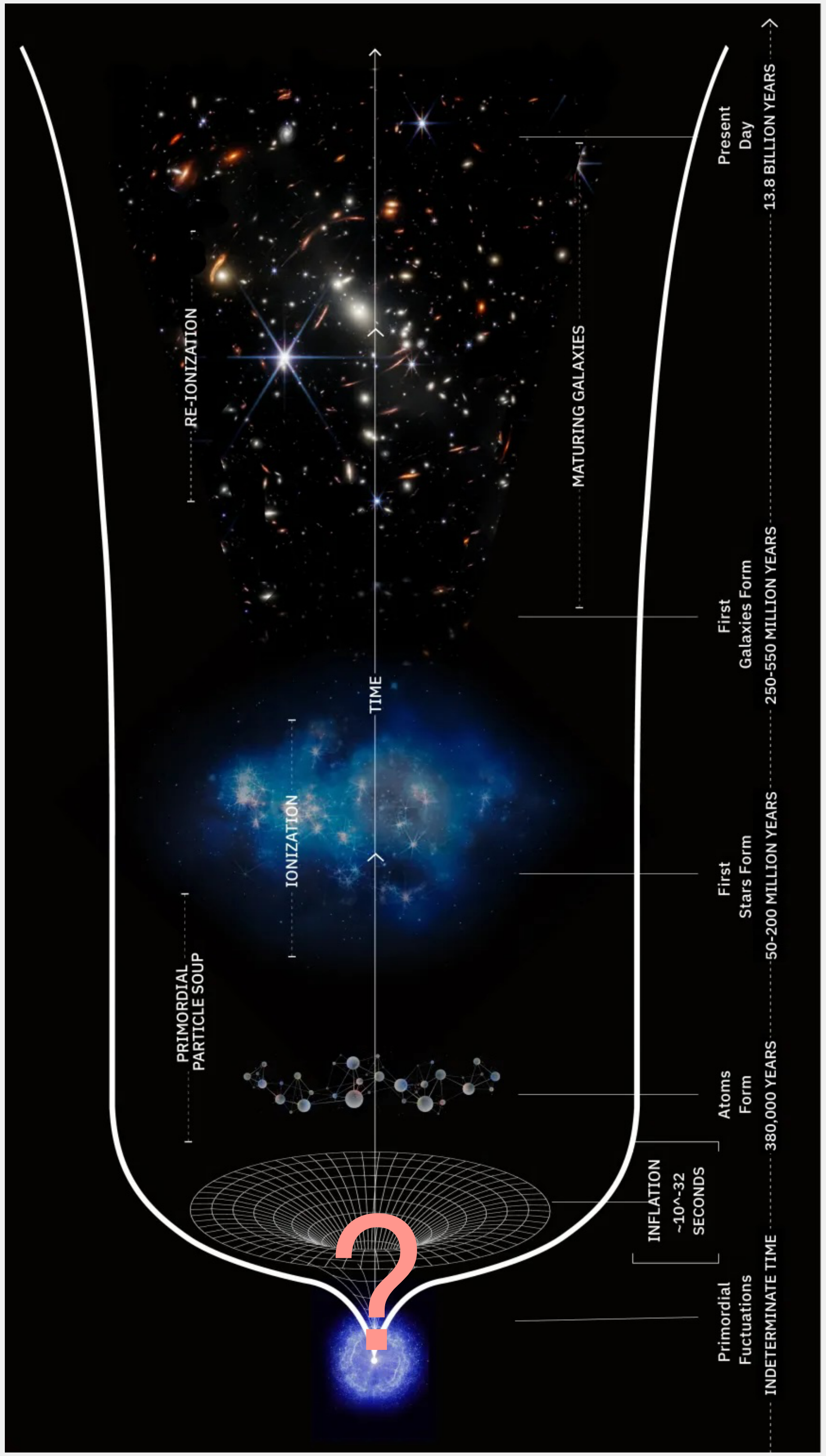
University of Waterloo

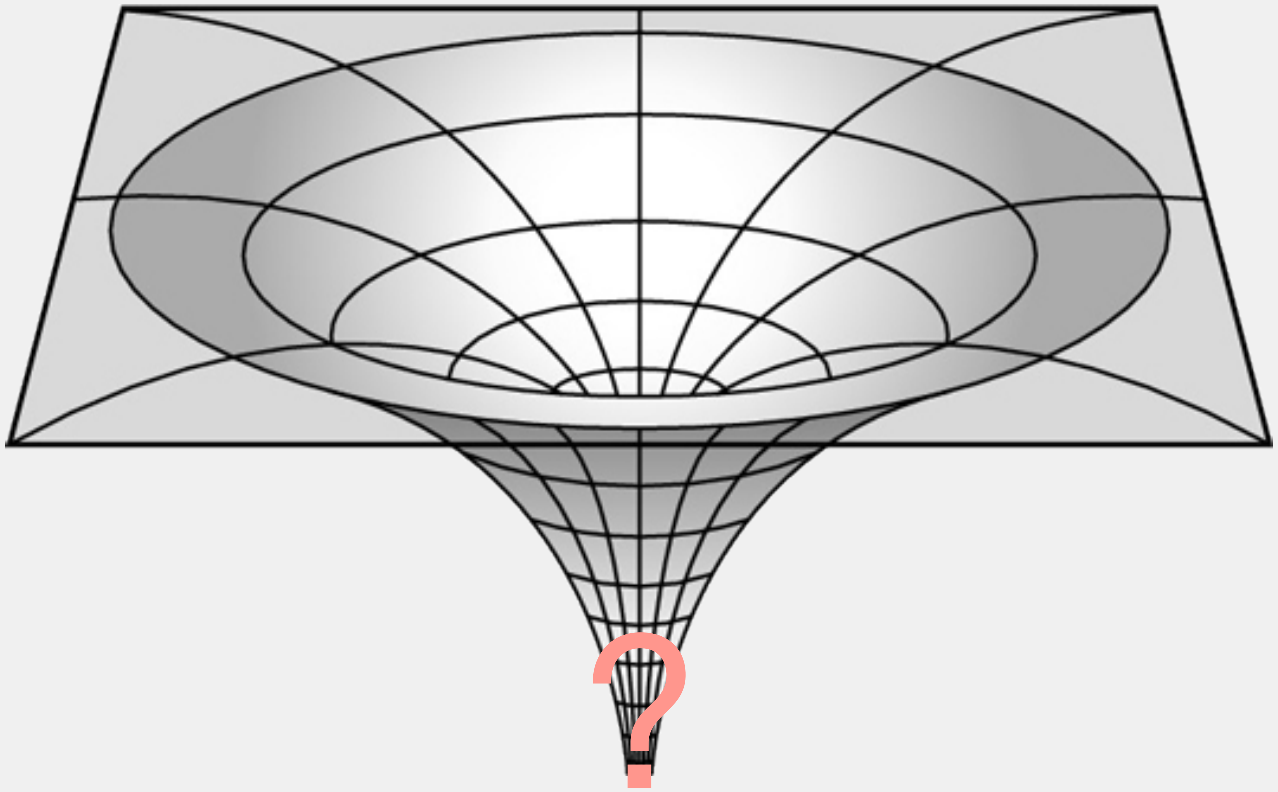
Perimeter Institute for Theoretical Physics

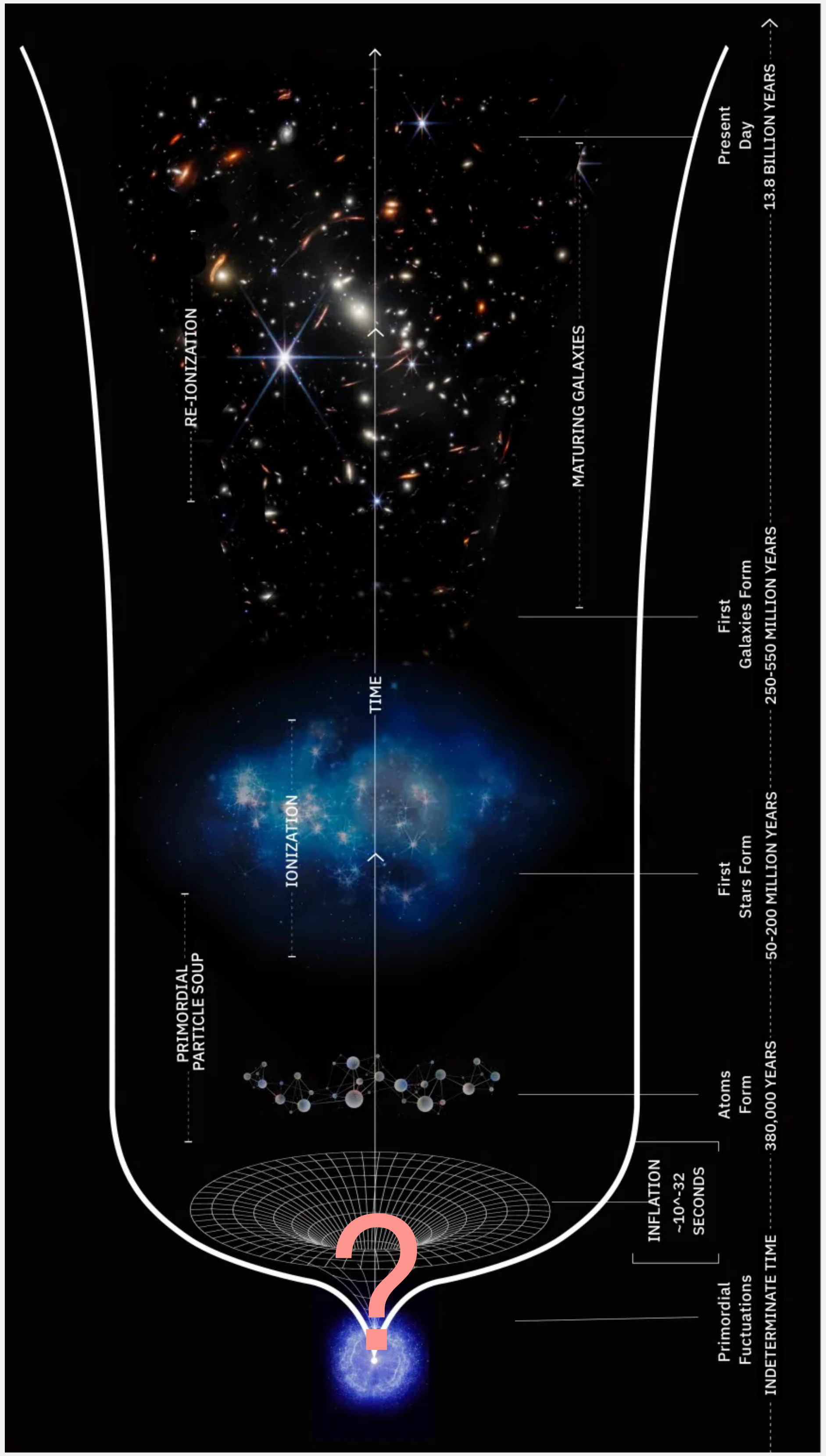
Brown U., BTPC IDEA Seminar

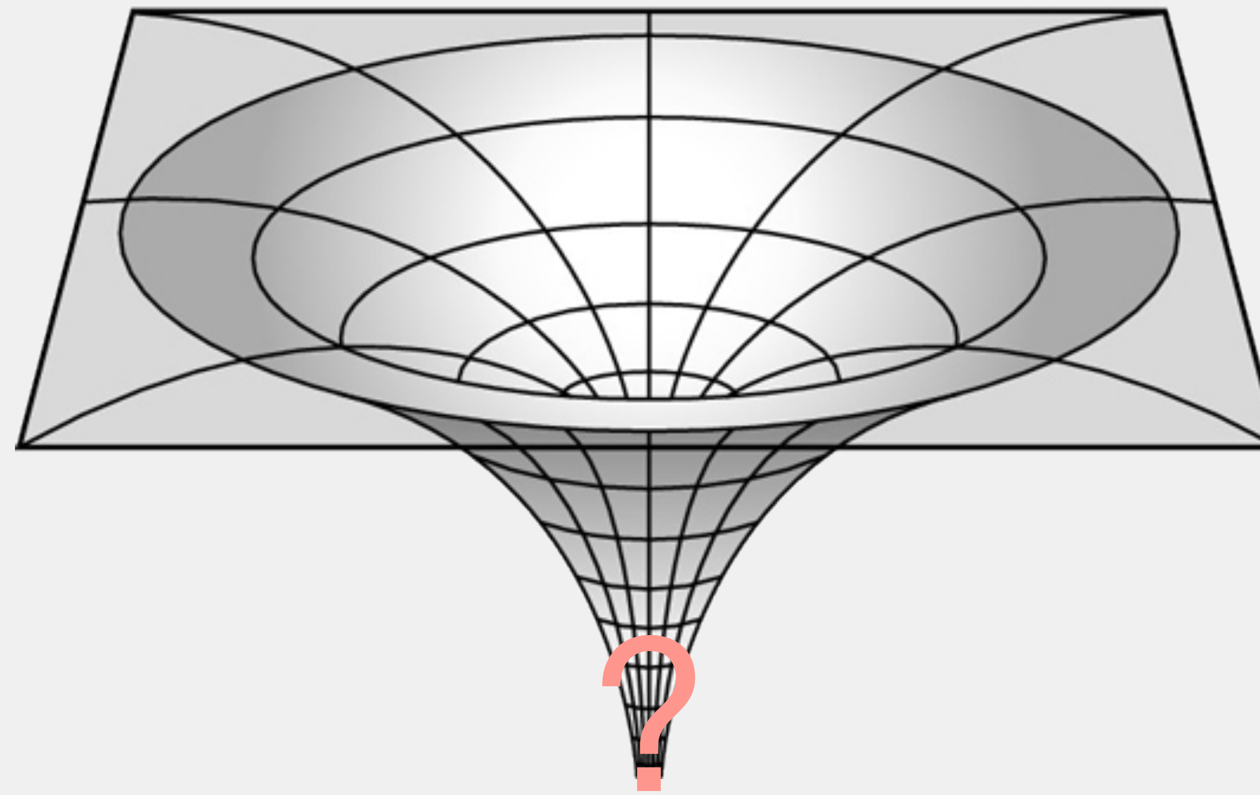
April 23rd, 2025

PERIMETER INSTITUTE

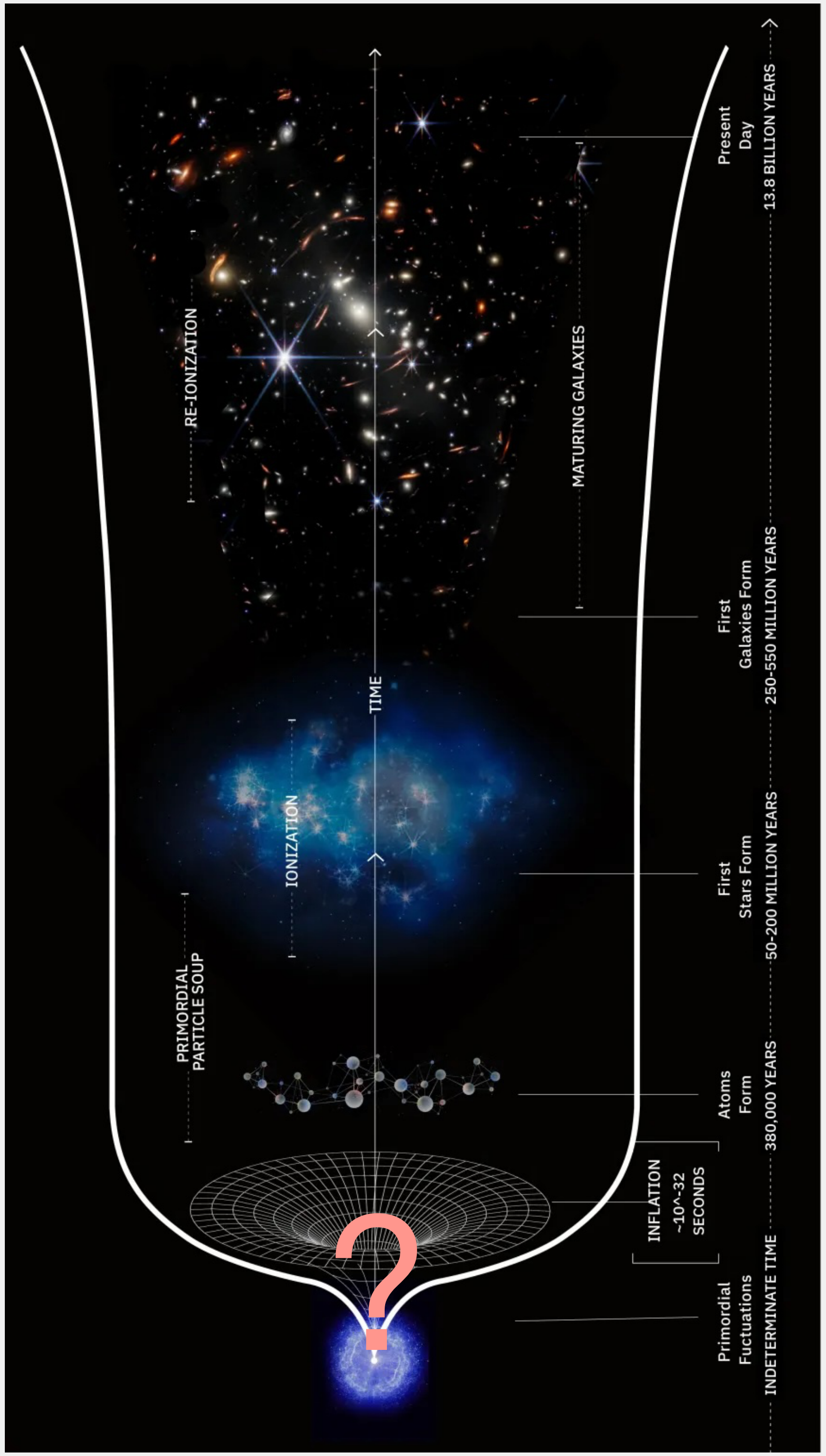

Present Day
13.8 BILLION YEARS
RE-IONIZATION
MATURING GALAXIES
TIME
First Galaxies Form
250-550 MILLION YEARS
IONIZATION
First Stars Form
50-200 MILLION YEARS
PRIMORDIAL PARTICLE SOUP
Atoms Form
380,000 YEARS
INFLATION ~10^-32 SECONDS
Primordial Fuctuations
INDETERMINATE TIME
?

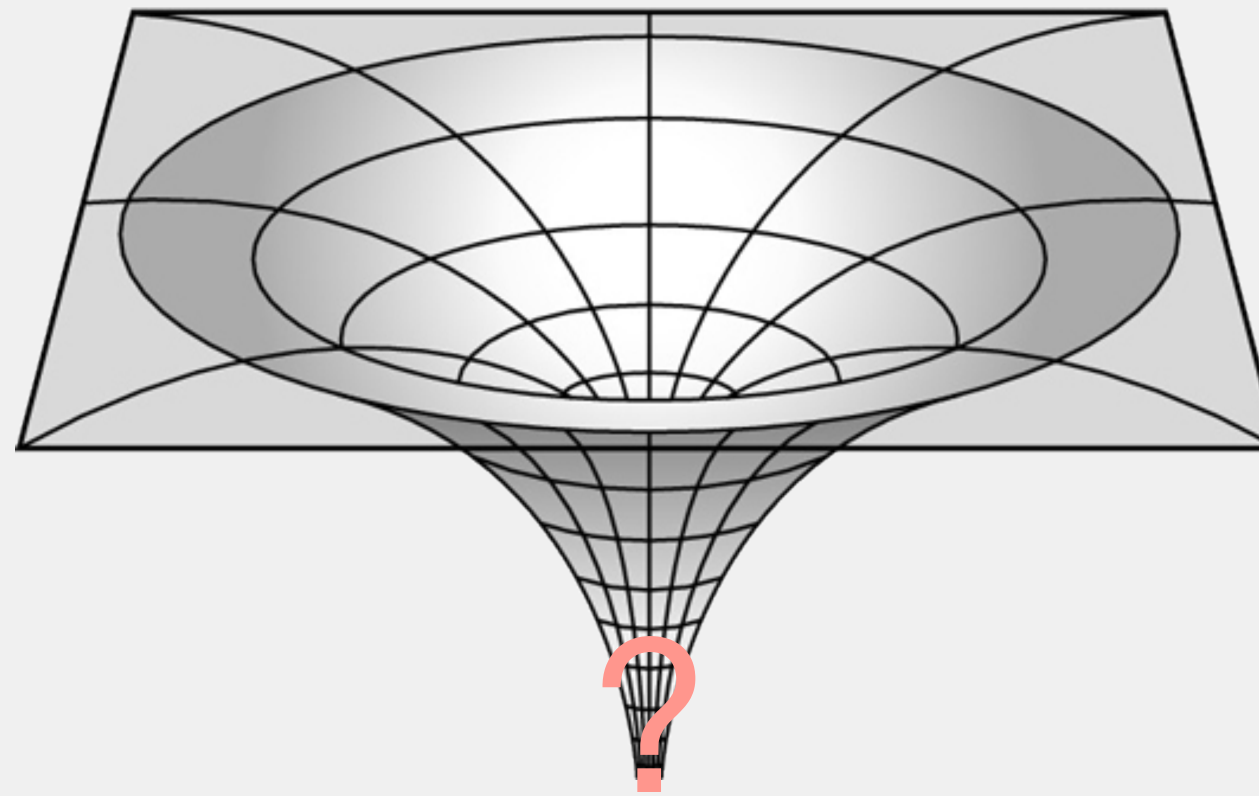

?





Part I: Lehners-JQ [JCAP01(2025)027]
Part IIa: Jonas-Lehners-JQ [JHEP08(2022)284]
Part IIb: Lehners-JQ [PLB 850(2024)138488]





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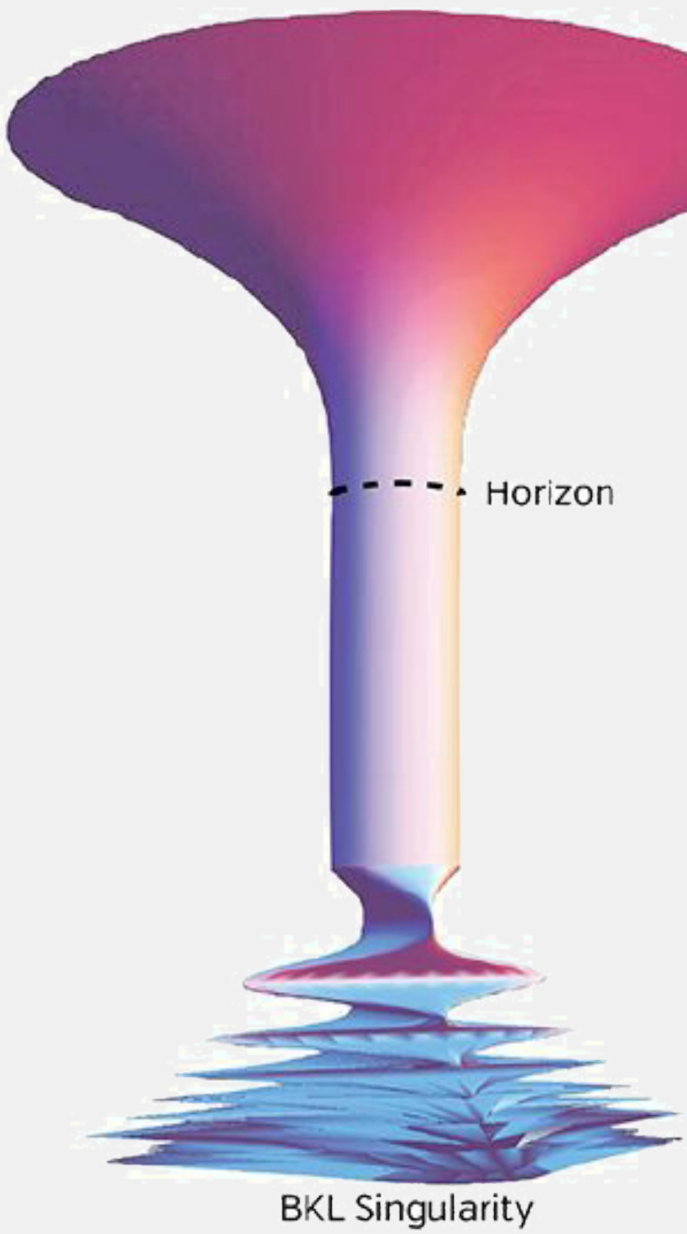
Part IIb: Lehners-JQ [PLB 850(2024)138488]

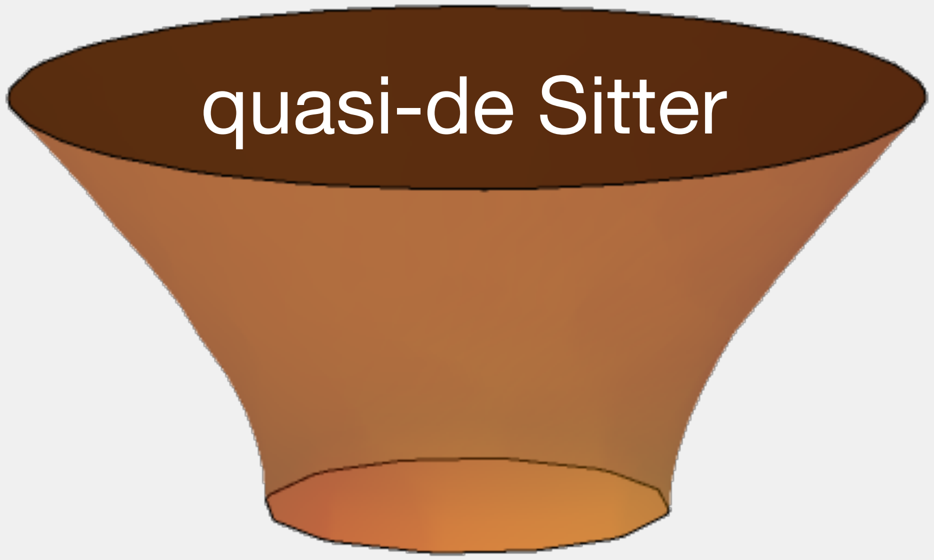
Part III: Liu-JQ-Afshordi [PRD 111(2025)044031]

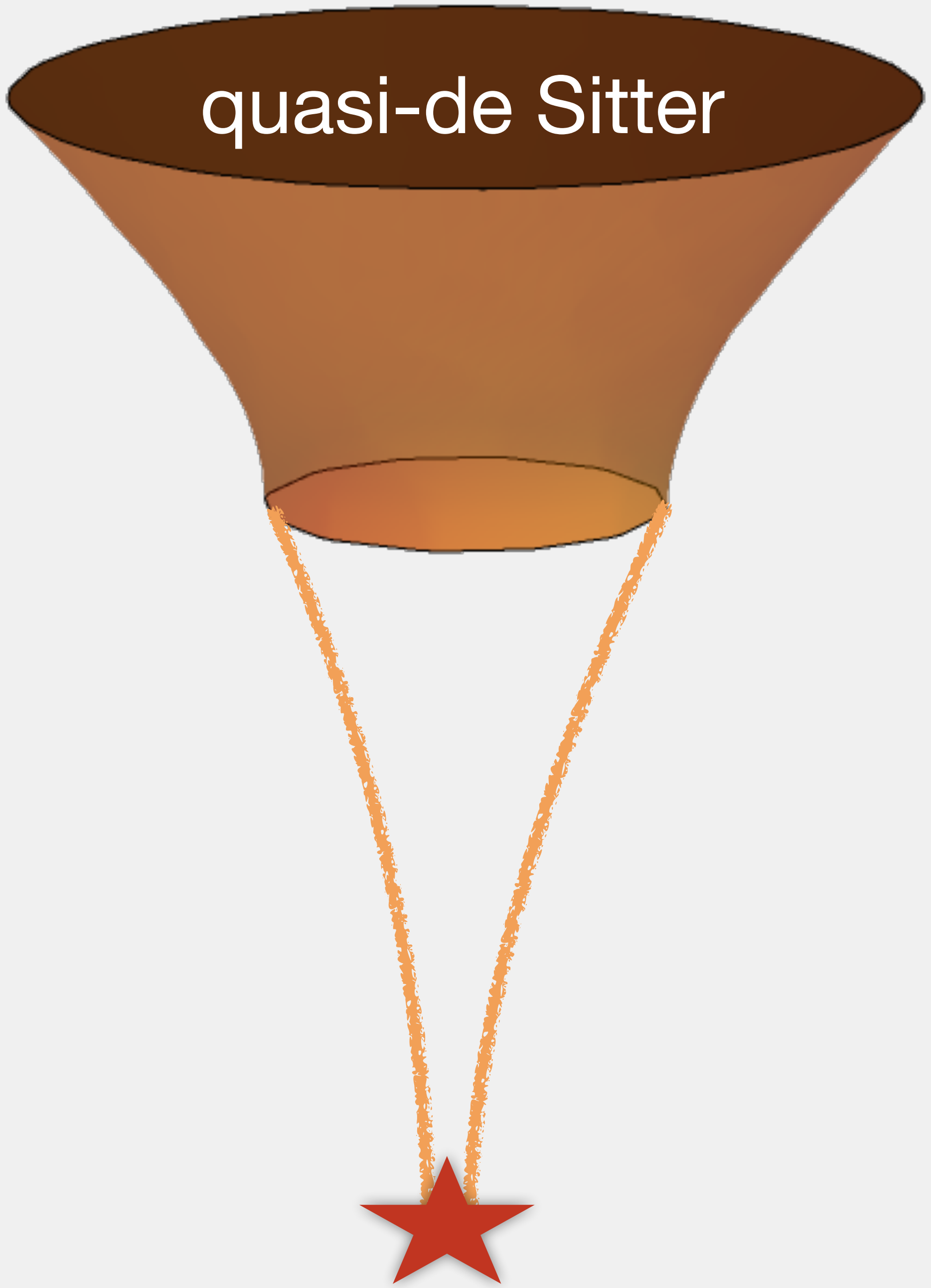
Intro

Complex metrics: why, what, how?

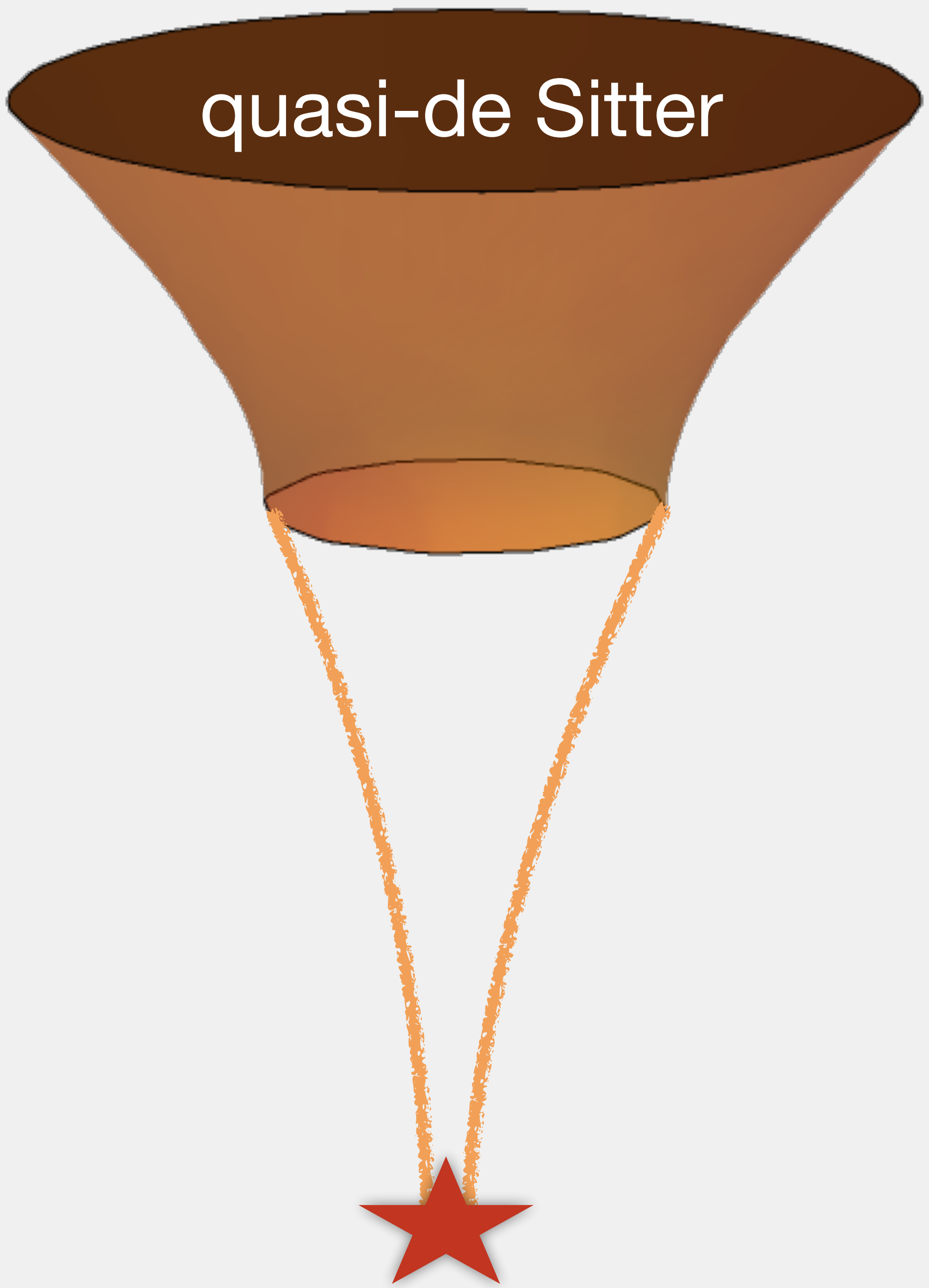


From Kip Thorne's
"The Science of Interstellar"


quasi-de Sitter

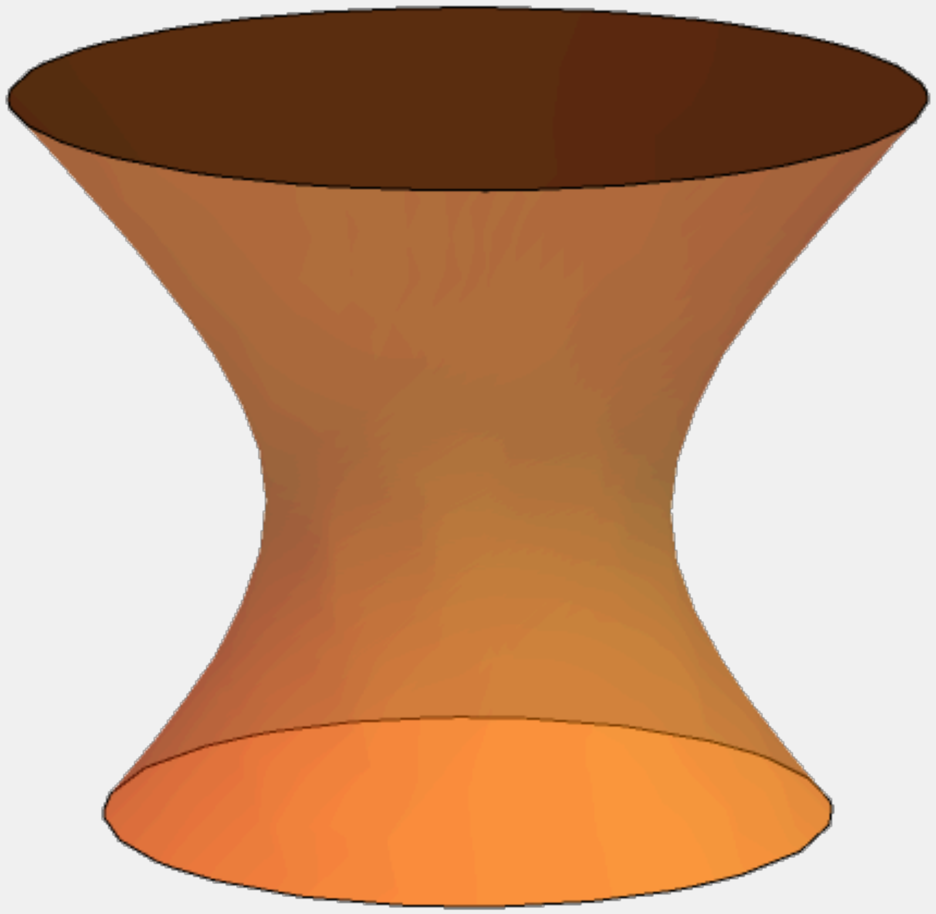


Borde-Guth-Vilenkin ‘03

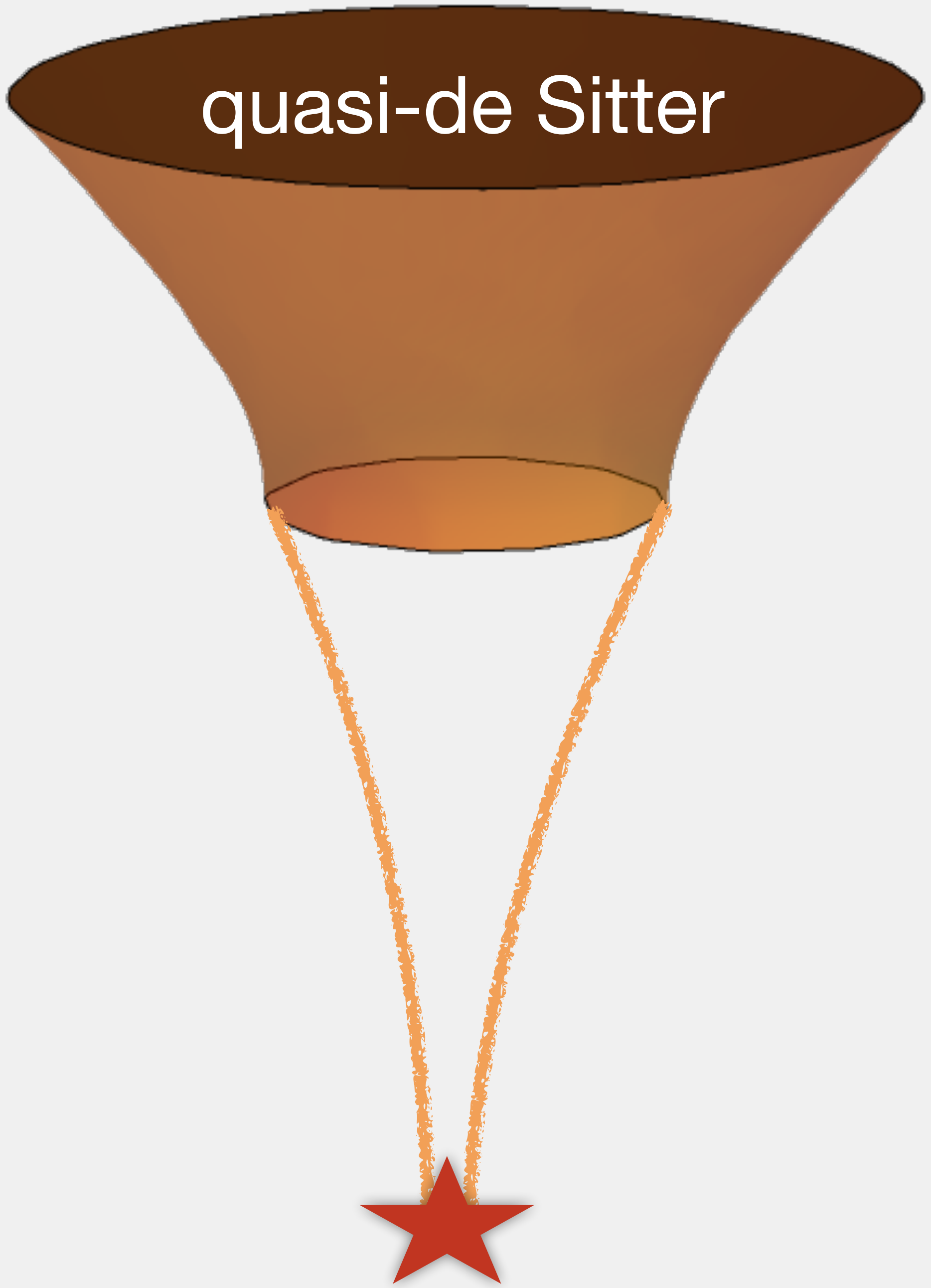


Borde-Guth-Vilenkin ‘03

classical
prelude
(extension)



Yoshida-JQ [CQG 35(2018)155019]
Geshnizjani-Ling-JQ [JHEP10(2023)182]



Borde-Guth-Vilenkin ‘03

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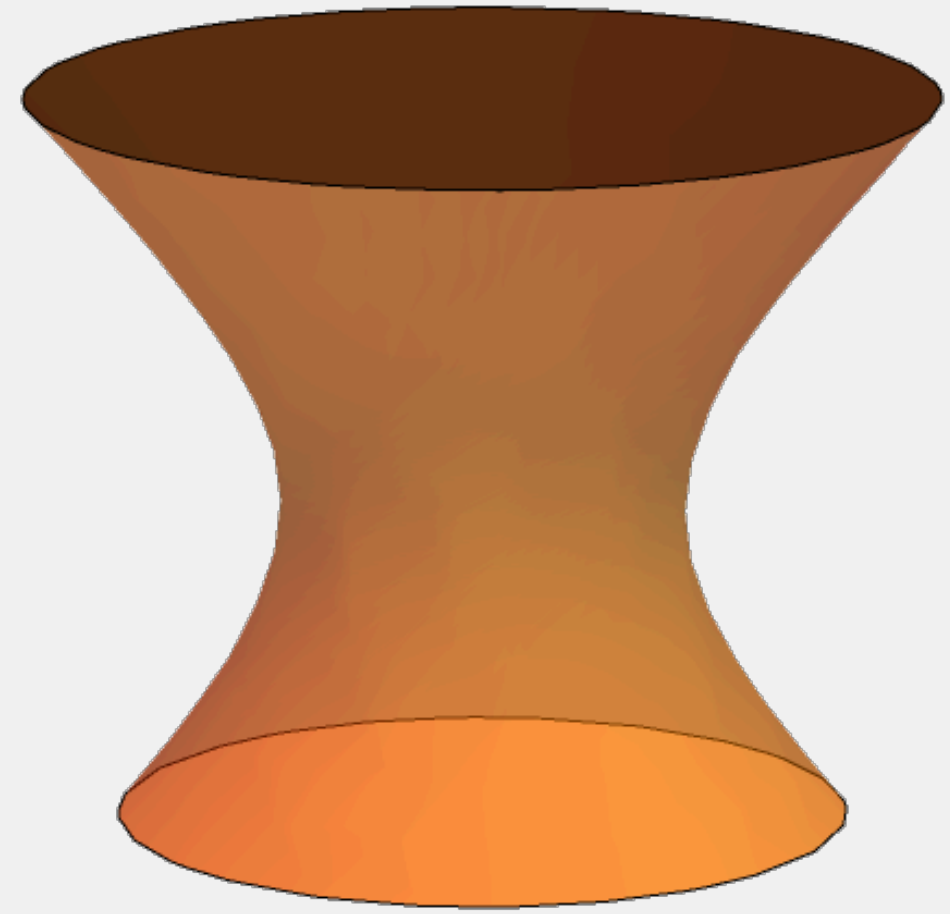
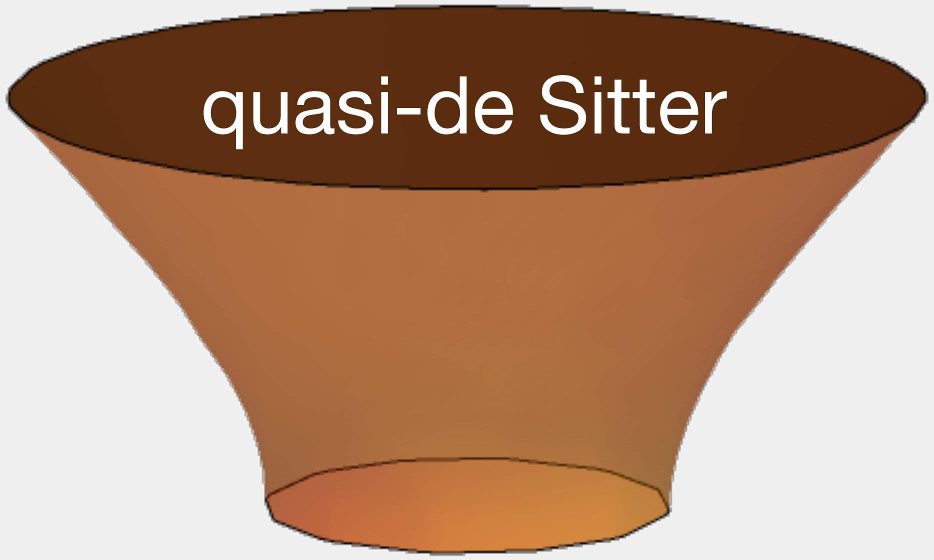
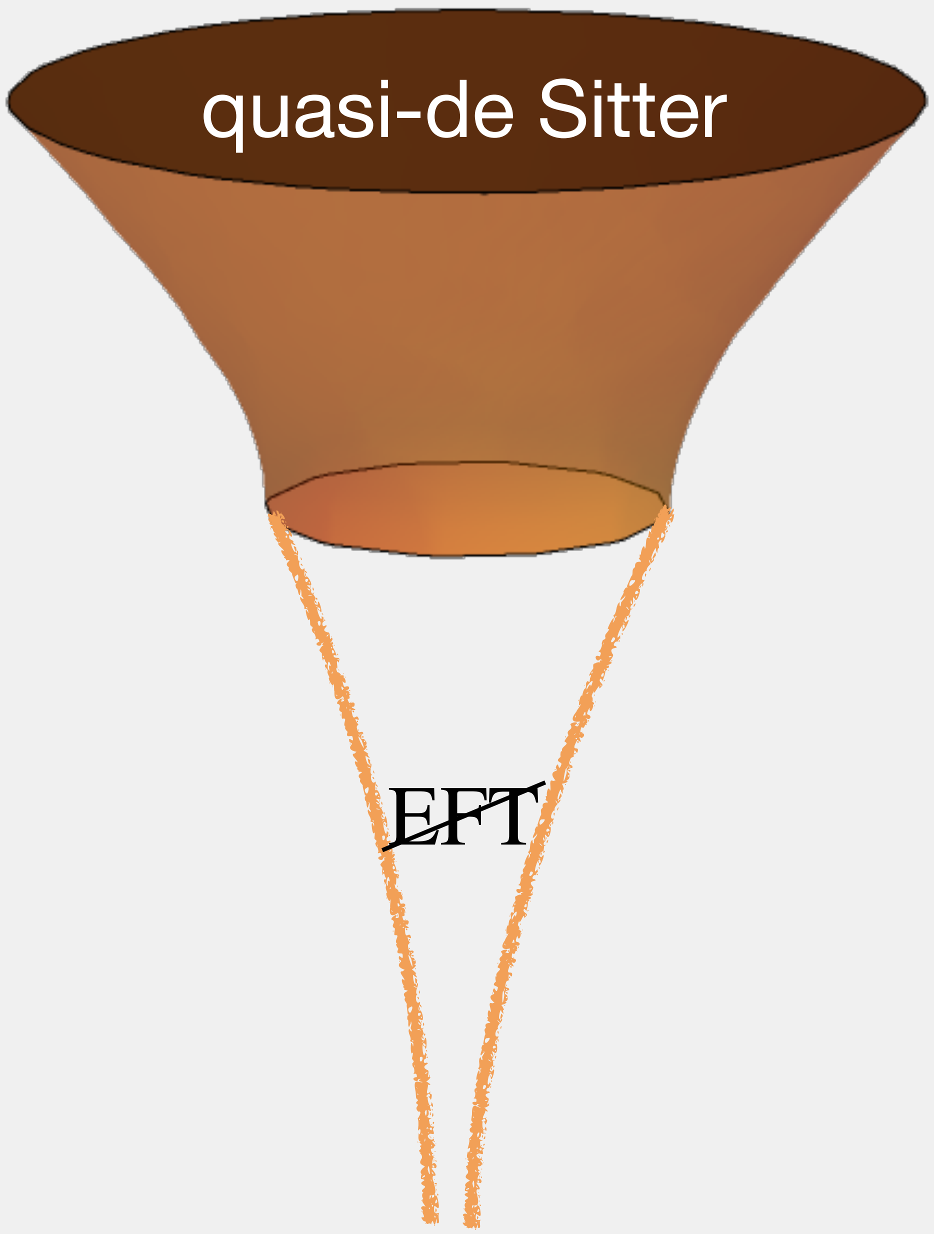


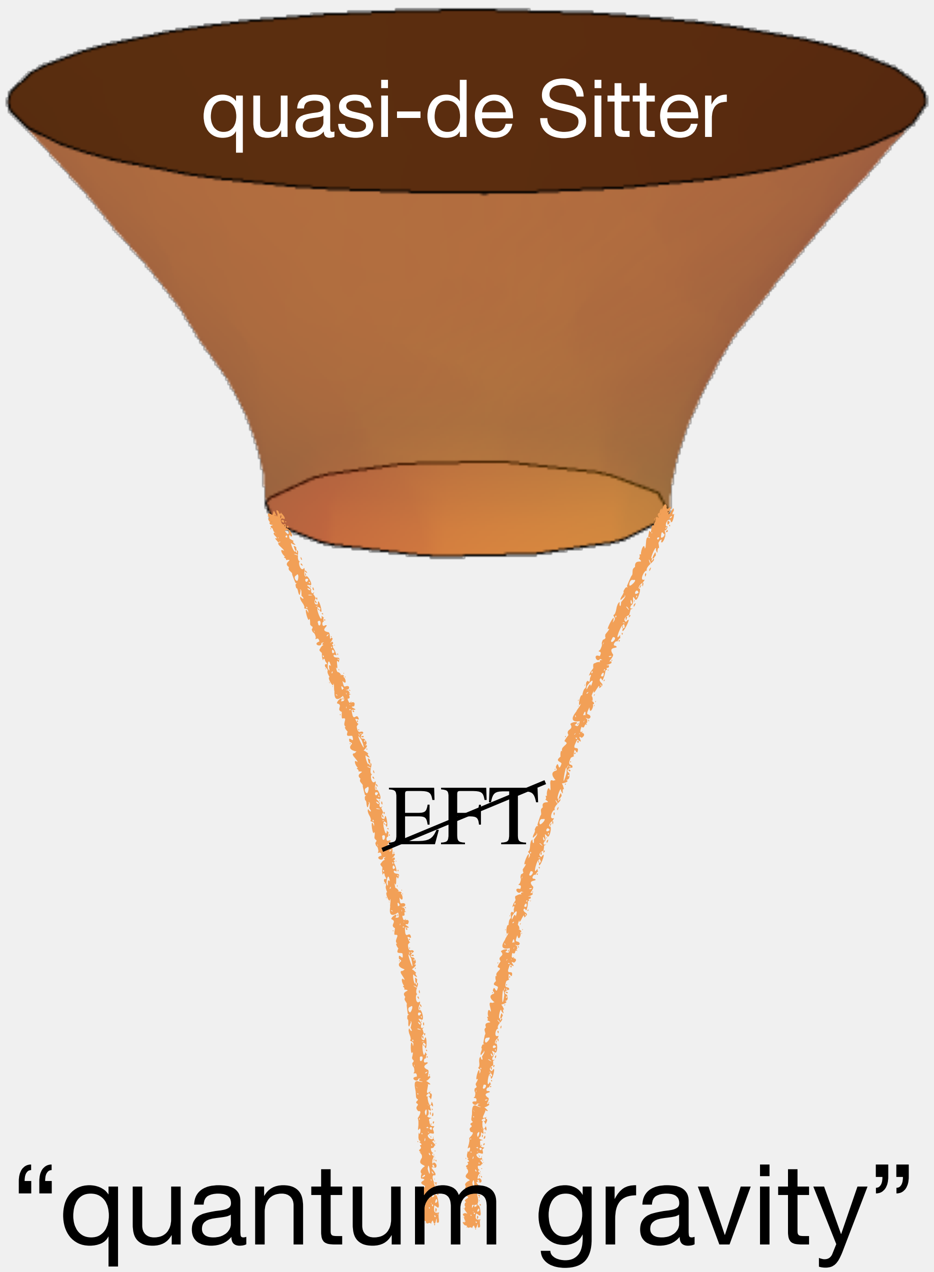
Yoshida-JQ [CQG 35(2018)155019]
Geshnizjani-Ling-JQ [JHEP10(2023)182]

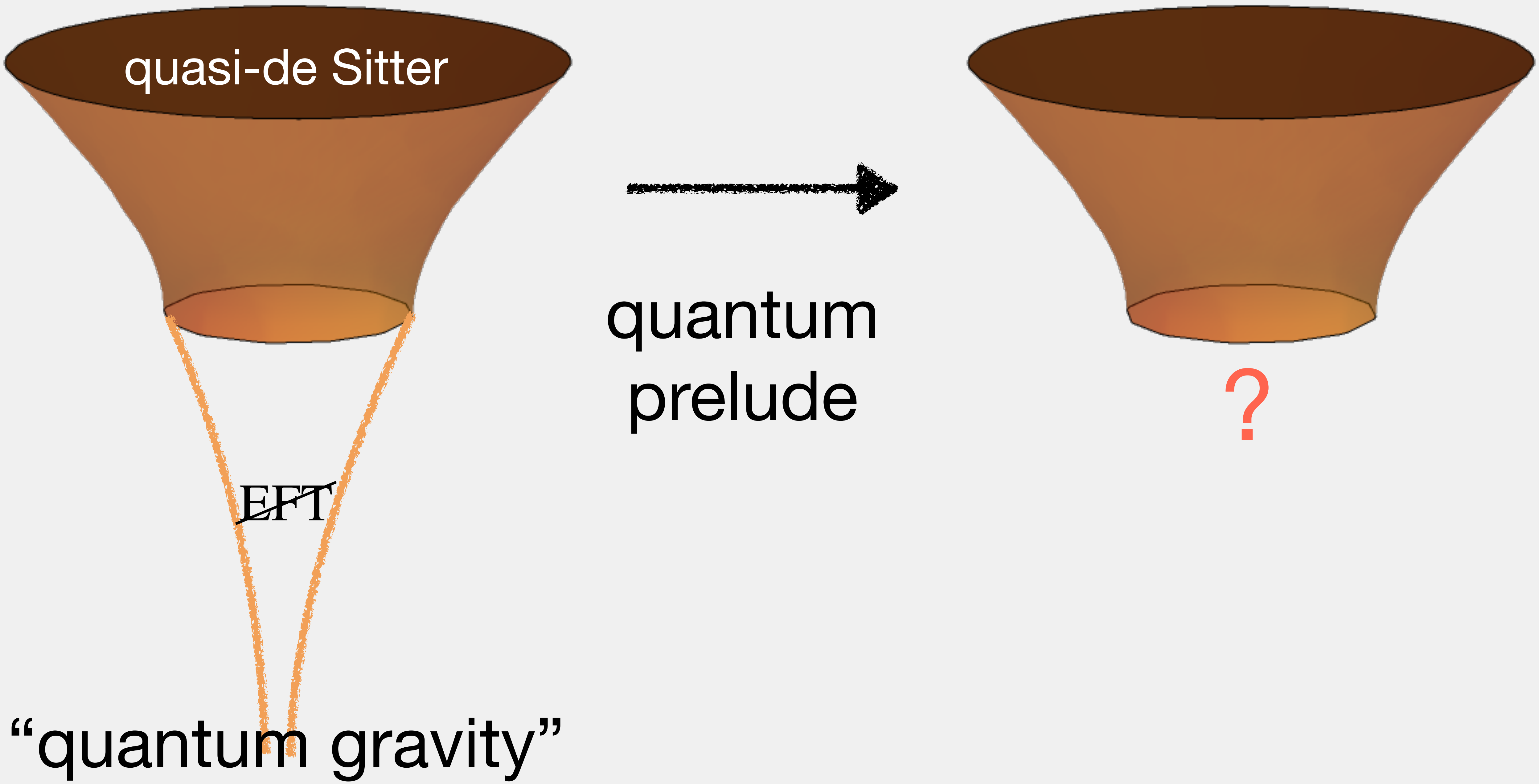
$$\text{flat FLRW}: \quad \lim_{t\to-\infty} a(t) = 0$$

$$\lim_{a\to 0^+} \frac{\mathrm{d}^\infty}{\mathrm{d}a^\infty}\left(\frac{H}{a}\frac{\mathrm{d}H}{\mathrm{d}a}\right) = \exists$$


quasi-de Sitter


quasi-de Sitter
EFT


quasi-de Sitter
~~EFT~~
“quantum gravity”


quasi-de Sitter
EFT
“quantum gravity”
quantum
prelude
?

Gravitational path integral

$$\Psi = \int_{\text{initial hypersurface}}^{\text{final hypersurface}} \mathscr{D}g \exp\left(\frac{i}{\hbar} S[g]\right)$$

Gravitational path integral

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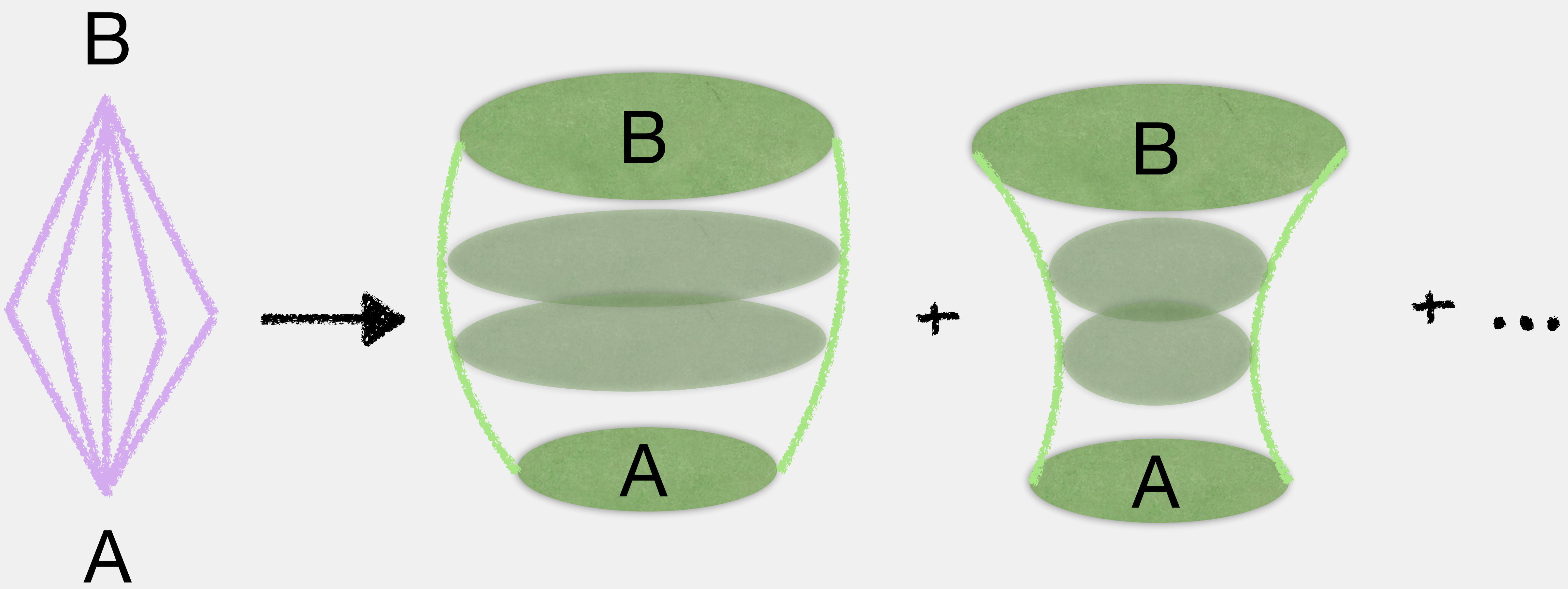


Gravitational path integral

$$\Psi = \int_{\mathsf{A}}^{\mathsf{B}} \mathscr{D}g \exp\left(\frac{i}{\hbar} S[g]\right) \sim \exp\left(\frac{i}{\hbar} S_{\text{on-shell}}[g_{\mathsf{A}\to\mathsf{B}}]\right) + \ldots$$

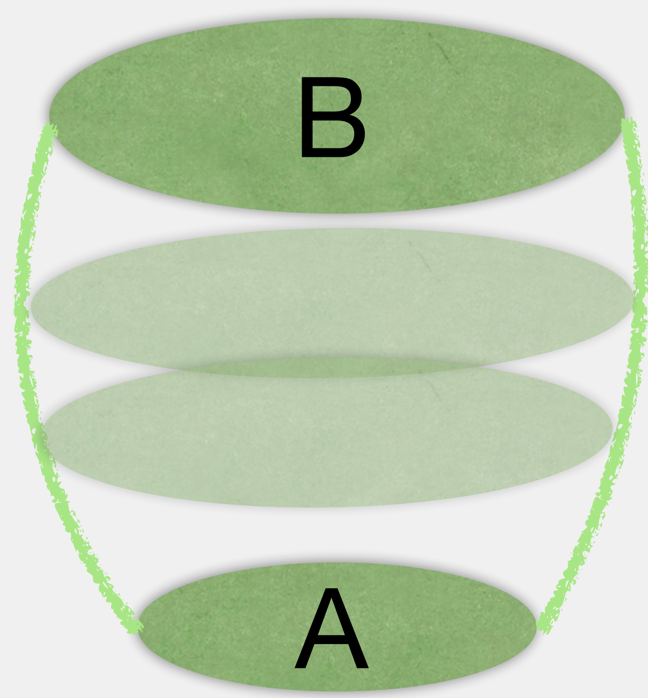
Gravitational path integral
as a sum over complex metrics

$g \in \mathbb{C}$

$$\Psi = \int_{\mathsf{A}}^{\mathsf{B}} \mathscr{D}g \exp\left(\frac{i}{\hbar}S[g]\right) \sim \exp\left(\frac{i}{\hbar}S_{\text{on-shell}}[g_{\mathsf{A}\to\mathsf{B}}]\right) + \ldots$$

$$\left.\frac{\delta S}{\delta g}\right|_{g=g_{\mathsf{A}\to\mathsf{B}}} = 0$$

$\Psi =$



$+ \ldots$

$g_{\mathsf{A}\to\mathsf{B}}$ is a “gravitational instanton”

Gravitational path integral
as a sum over complex metrics

$$\Psi = \int_{\mathsf{A}}^{\mathsf{B}} \mathscr{D}g \exp\left(\frac{i}{\hbar}S[g]\right) \sim \exp\left(\frac{i}{\hbar}S_{\text{on-shell}}[g_{\mathsf{A}\to\mathsf{B}}]\right) + \ldots$$

$g \in \mathbb{C}$

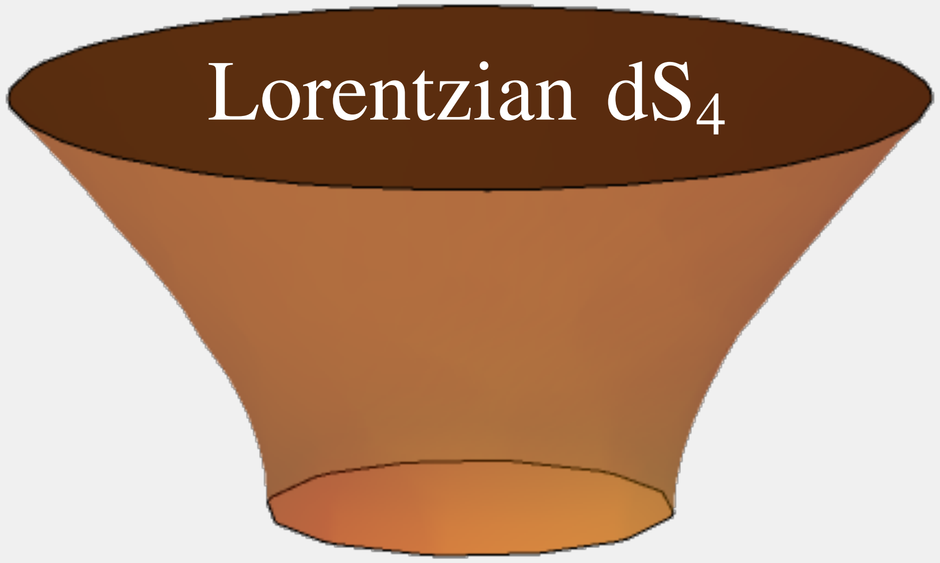
$$\left.\frac{\delta S}{\delta g}\right|_{g=g_{\mathsf{A}\to\mathsf{B}}} = 0$$

$$\Psi \sim \exp\left(\frac{i}{\hbar}S_{\text{on-shell}}[g_{\mathsf{A}\to\mathsf{B}}]\right) = \exp\left(\frac{1}{\hbar}\left(\mathscr{W} + i\mathscr{S}\right)\right), \qquad \mathscr{W}, \mathscr{S} \in \mathbb{R}$$

$$\implies \mathscr{S} \approx \operatorname{Re} S_{\text{on-shell}}, \quad \mathscr{W} \approx -\operatorname{Im} S_{\text{on-shell}}$$

$$|\Psi|^2 \sim \exp\left(\frac{2\mathscr{W}}{\hbar}\right) \longrightarrow \text{probability density}$$

$$g = -\mathrm{d}t^2 + \cosh^2(t)\mathrm{d}\Omega^2_{(3)}$$



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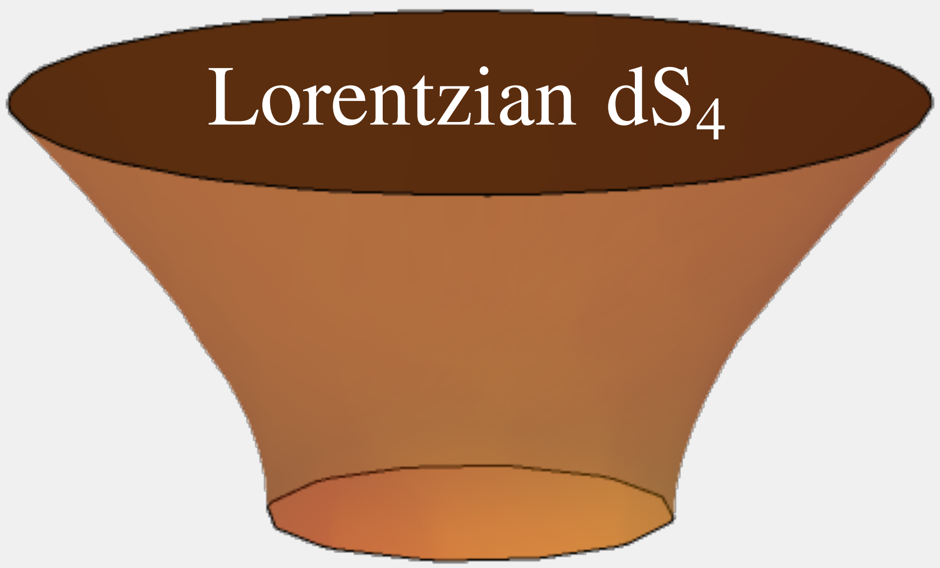
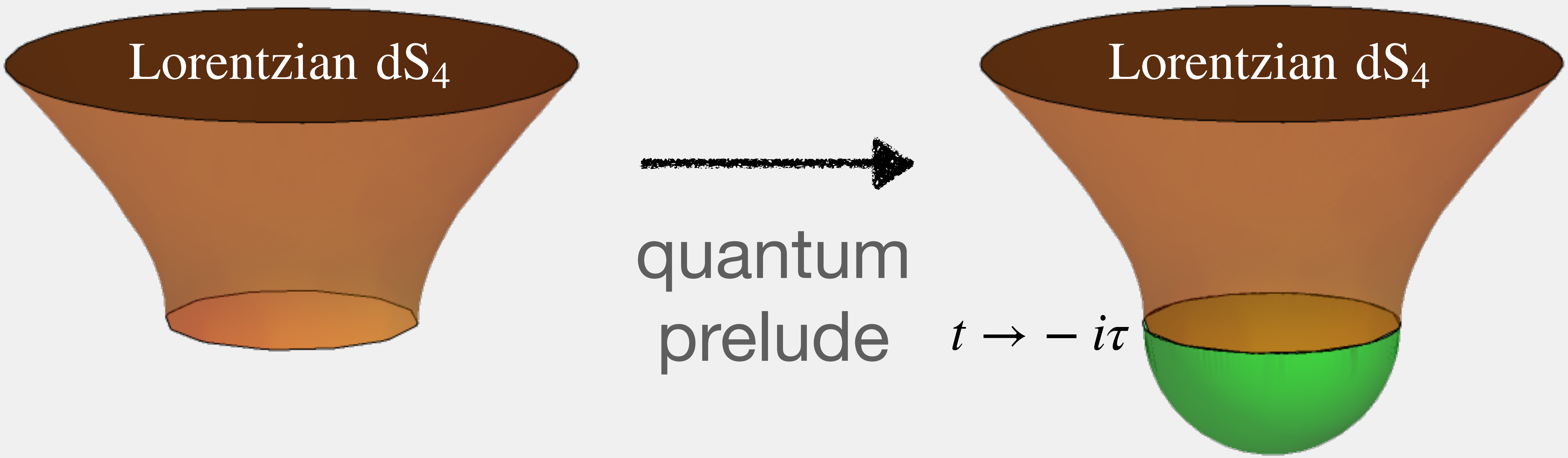


quantum
prelude

$$g = -\,\mathrm{d}t^2 + \cosh^2(t)\mathrm{d}\Omega^2_{(3)}$$



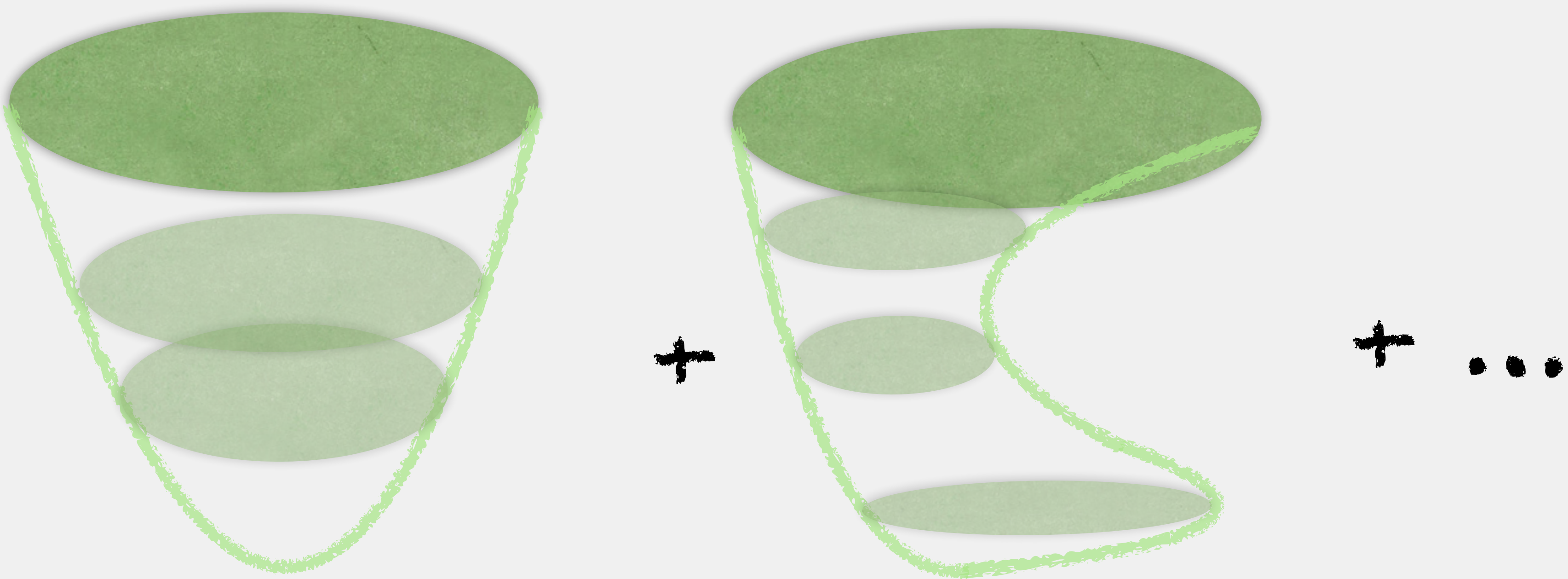
Euclidean S^4

$$g = \mathrm{d}\tau^2 + \sin^2(\tau)\mathrm{d}\Omega^2_{(3)}$$

Hartle-Hawking **no-boundary** proposal

Hartle-Hawking wave function

$$\Psi = \int_{\text{no boundary}}^{\text{final hypersurface}} \mathscr{D}g \, e^{-S_{\mathrm{E}}[g]/\hbar}$$



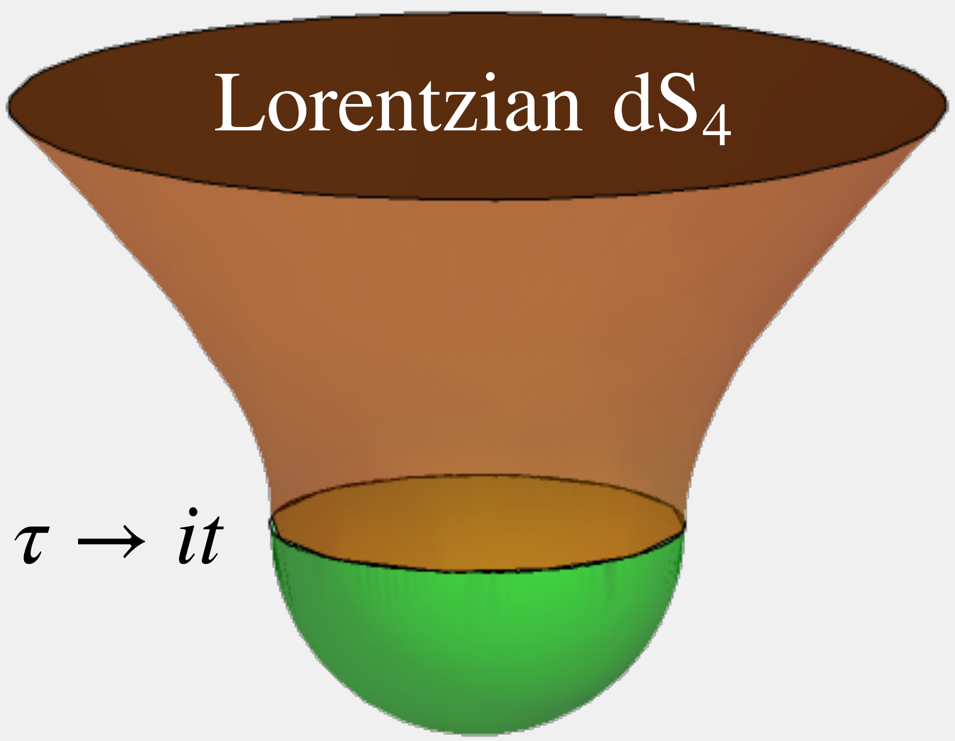
no boundary = compact and regular spacetime $\Longrightarrow$ closed universe

Hartle-Hawking wave function

Einstein gravity + Λ

$H = \sqrt{\Lambda/3}$

$$g = -\,\mathrm{d}t^2 + H^{-2}\cosh^2(Ht)\,\mathrm{d}\Omega^2_{(3)}$$



Euclidean S^4

$$g = \mathrm{d}\tau^2 + H^{-2}\sin^2(H\tau)\,\mathrm{d}\Omega^2_{(3)}$$

$8\pi G_N = c = 1$

Hartle-Hawking wave function

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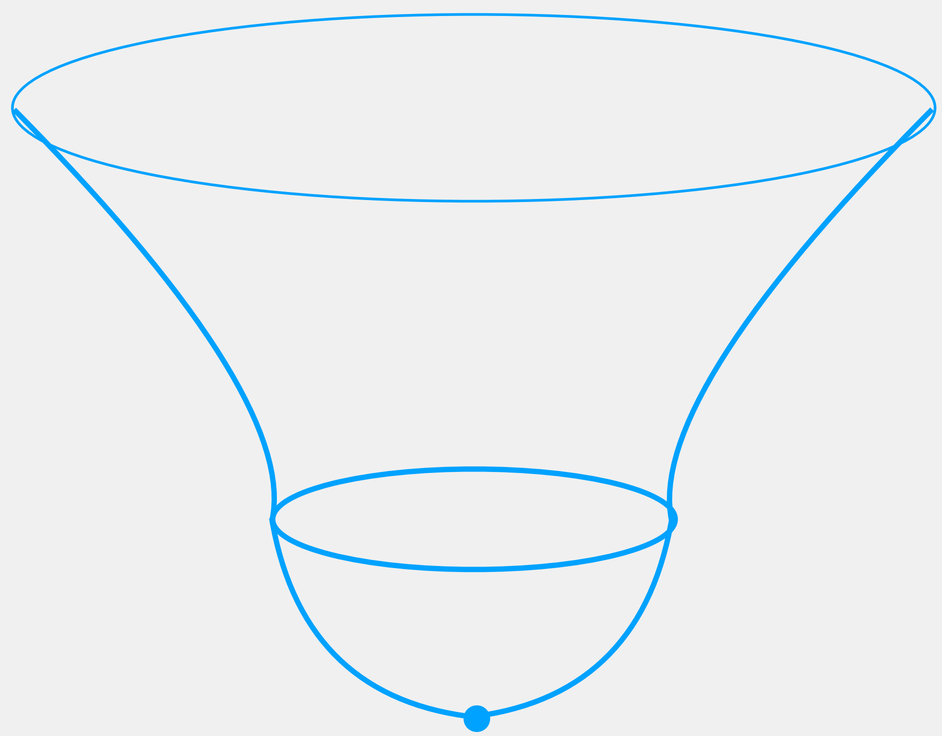

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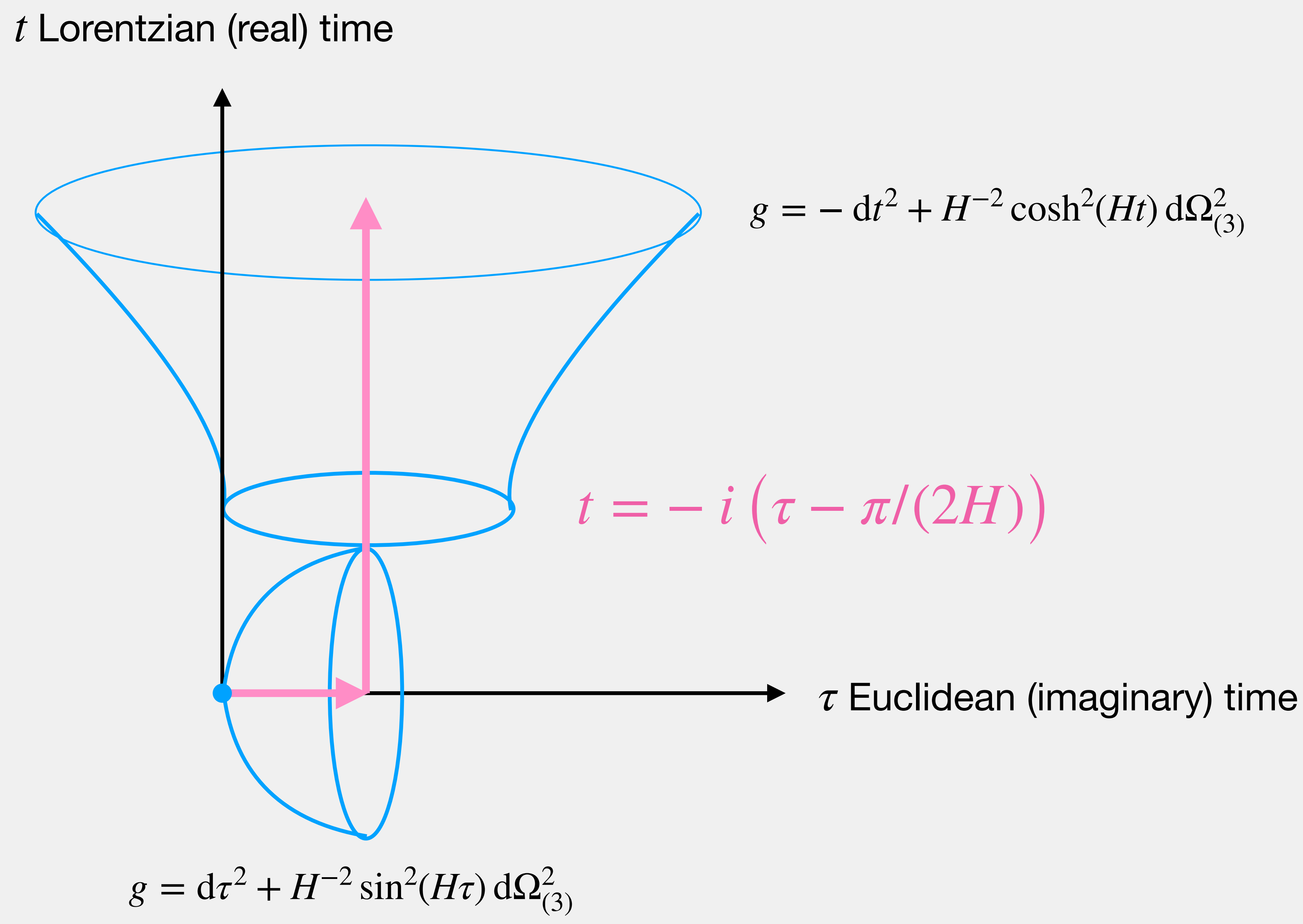

$8\pi G_{\mathrm{N}} = c = 1$

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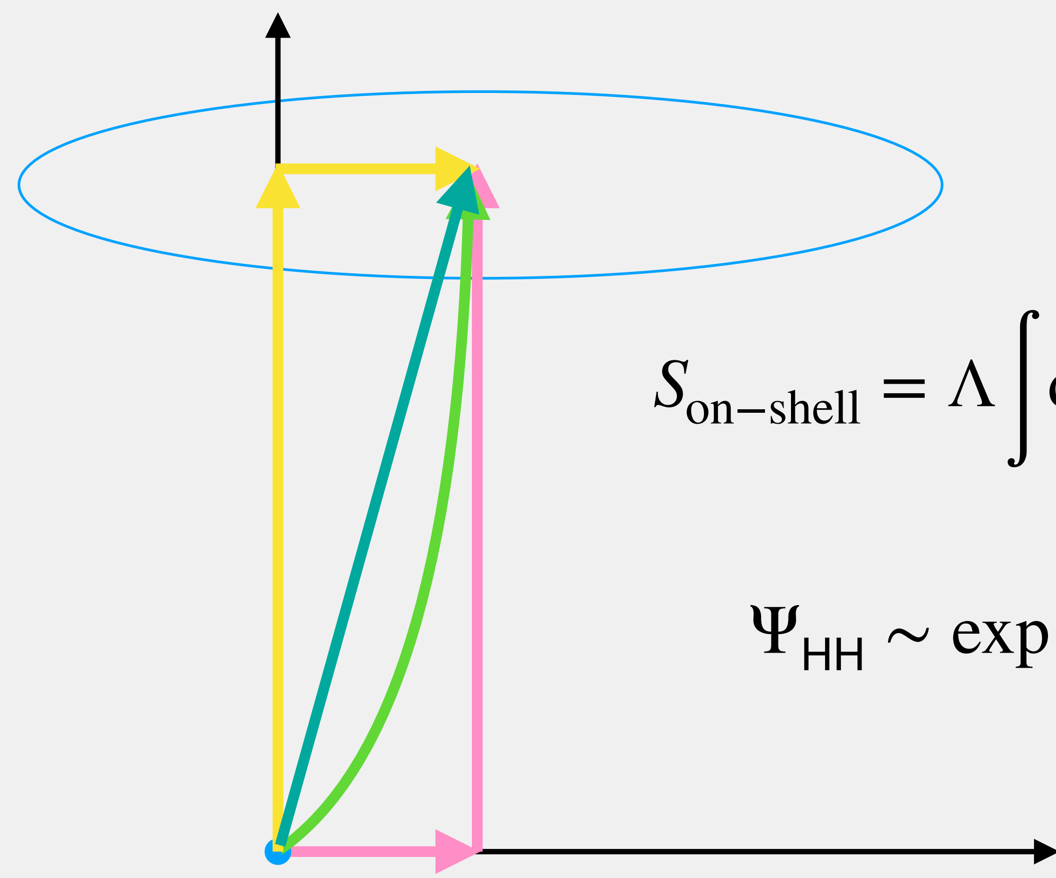
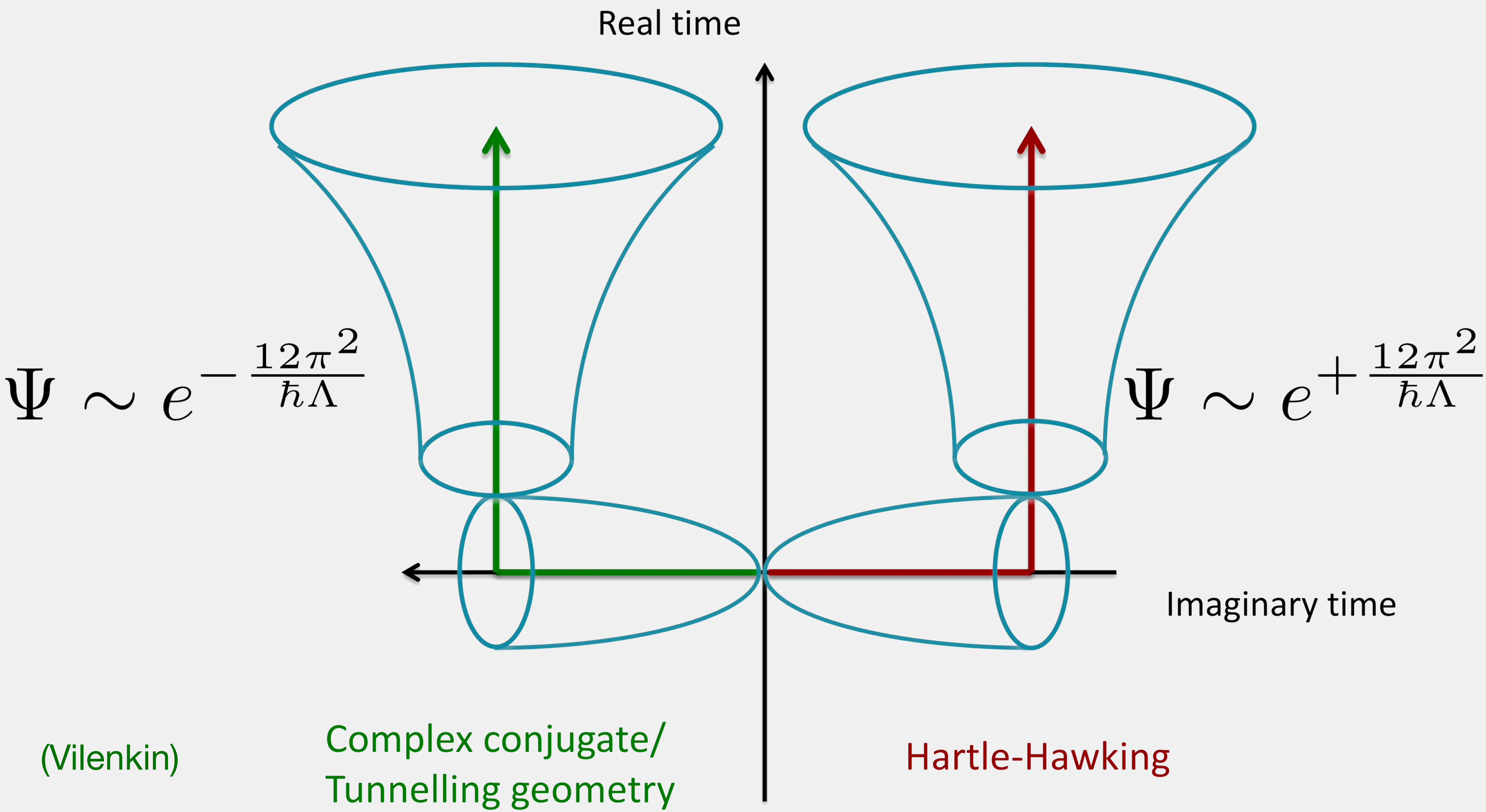
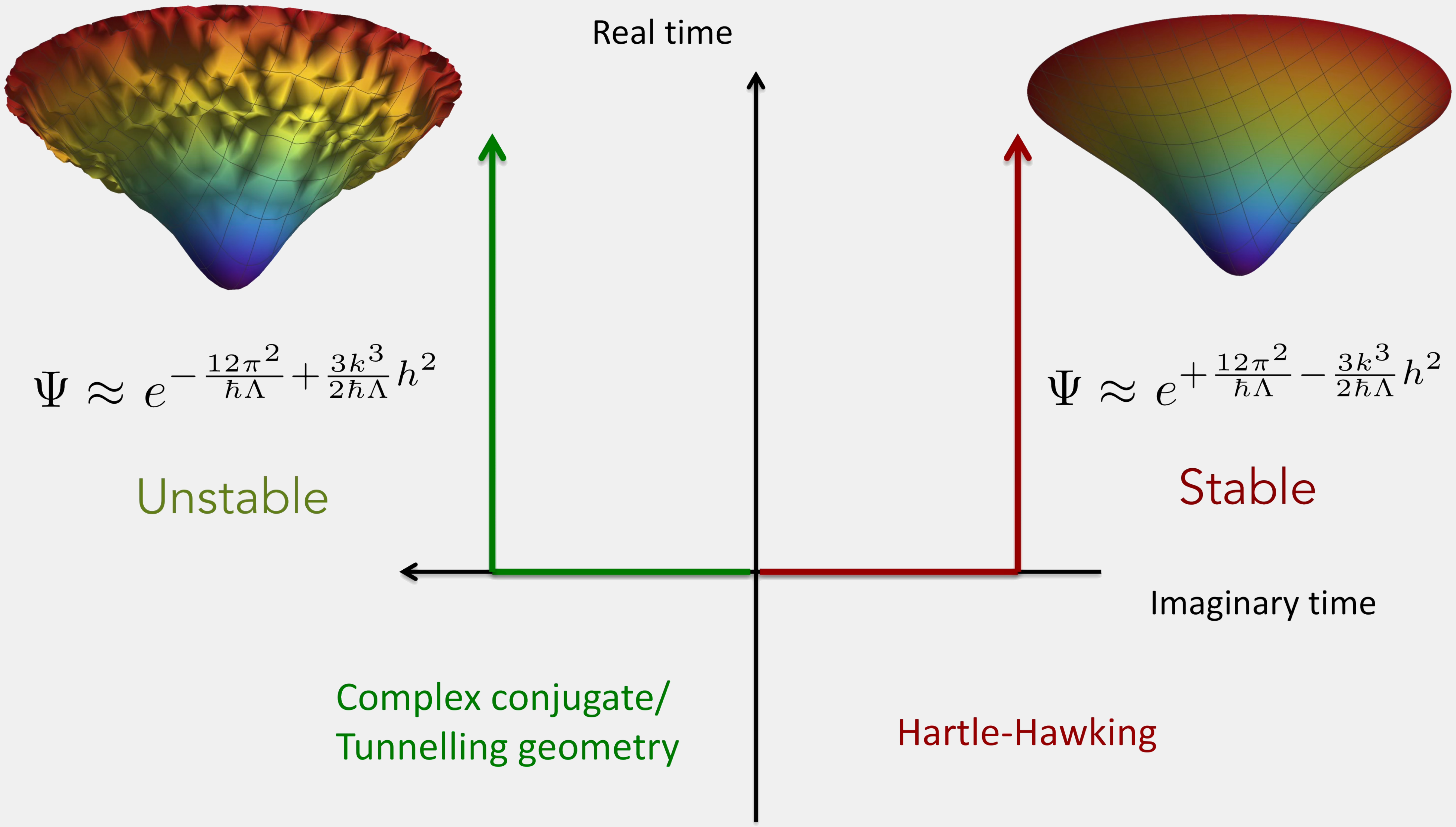


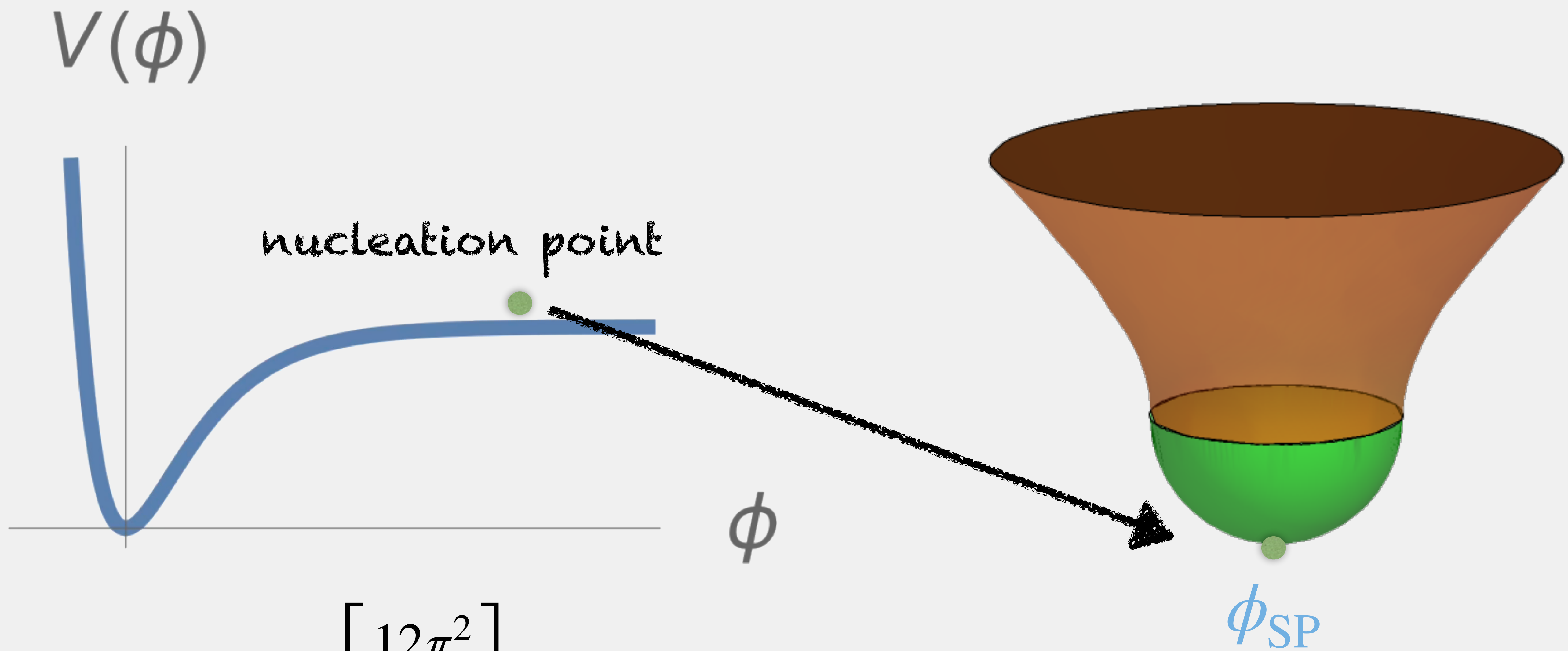
$$S_{\mathrm{on-shell}} = \Lambda \int \mathrm{d}^4 x \sqrt{|\det(g_{\mu\nu})|} \sim H^2 (H^{-1})^4 \sim H^{-2}$$

$$\Psi_{\mathrm{HH}} \sim \exp\left(\frac{12\pi^2}{\hbar\Lambda}\right)$$



Effectively opposite Wick rotations


Real time
$\Psi \approx e^{-\frac{12\pi^2}{\hbar\Lambda}+\frac{3k^3}{2\hbar\Lambda}h^2}$
Unstable
$\Psi \approx e^{+\frac{12\pi^2}{\hbar\Lambda}-\frac{3k^3}{2\hbar\Lambda}h^2}$
Stable
Imaginary time
Complex conjugate/
Tunnelling geometry
Hartle-Hawking

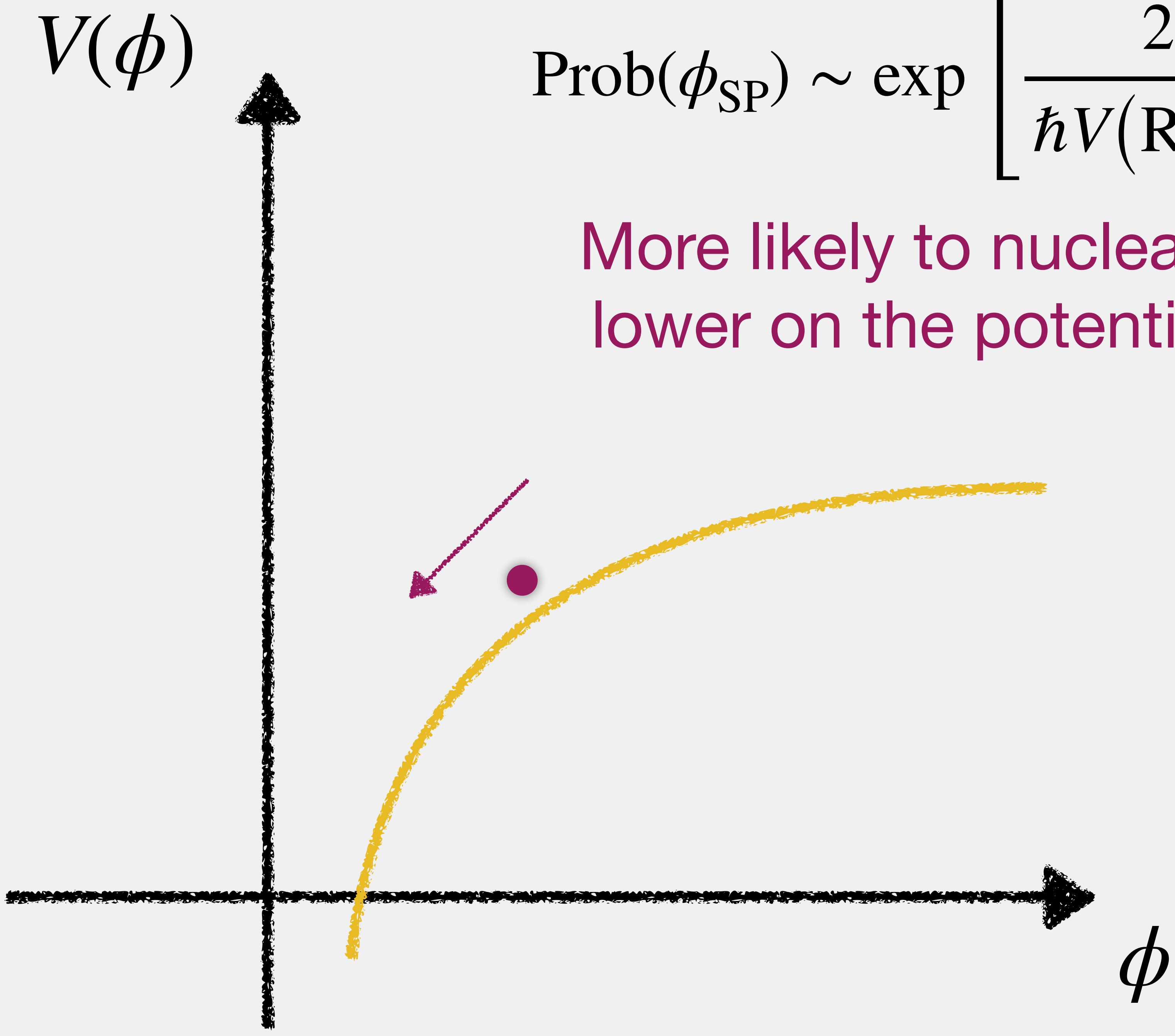


$$\Psi_{\rm HH} \sim \exp\left[\frac{12\pi^2}{\hbar\Lambda}\right]$$

$$\longrightarrow \quad \mathrm{Prob}(\phi_{\rm SP}) \sim \exp\left[\frac{24\pi^2}{\hbar V\big(\mathrm{Re}(\phi_{\rm SP})\big)}\right]$$

$$\text{Prob}(\phi_{\text{SP}}) \sim \exp\left[\frac{24\pi^2}{\hbar V\big(\text{Re}(\phi_{\text{SP}})\big)}\right]$$

More likely to nucleate lower on the potential



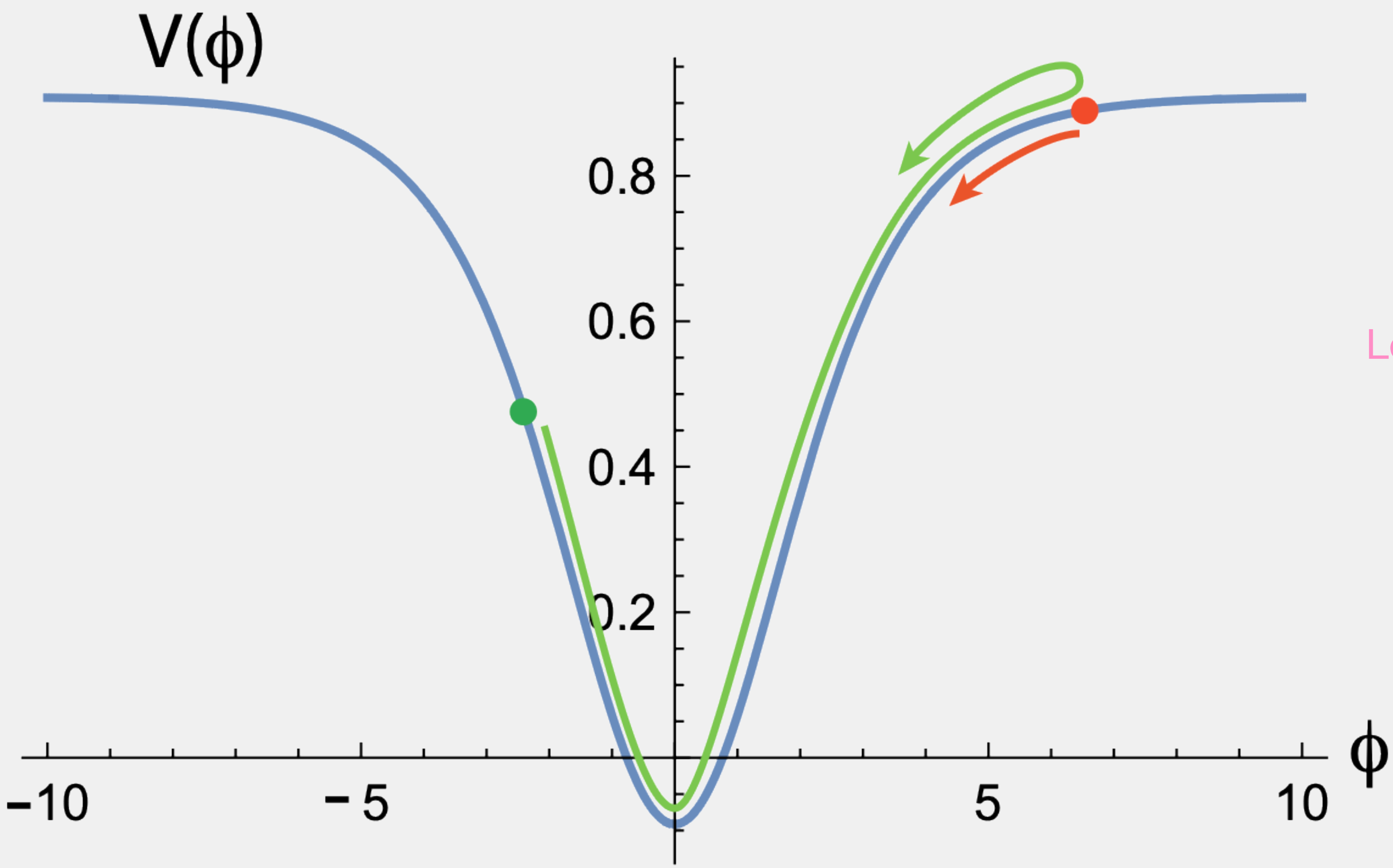
- The more difficult it becomes to get inflation
- The universe could collapse after nucleation

Part I

Is the Universe more likely to bounce before inflating?

Matsui-Takahashi-Terada '19, Matsui+ '23

$$V(\phi) = \alpha \tanh^2\left(\frac{\phi}{\sqrt{6}}\right) + \beta \tanh\left(\frac{\phi}{\sqrt{6}}\right) + \gamma$$



Lehners-JQ [JCAP01(2025)027]

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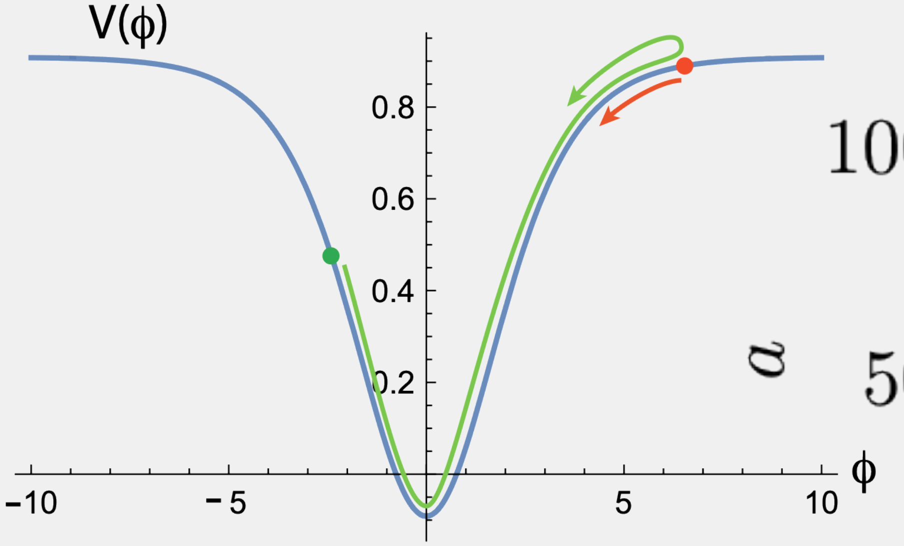
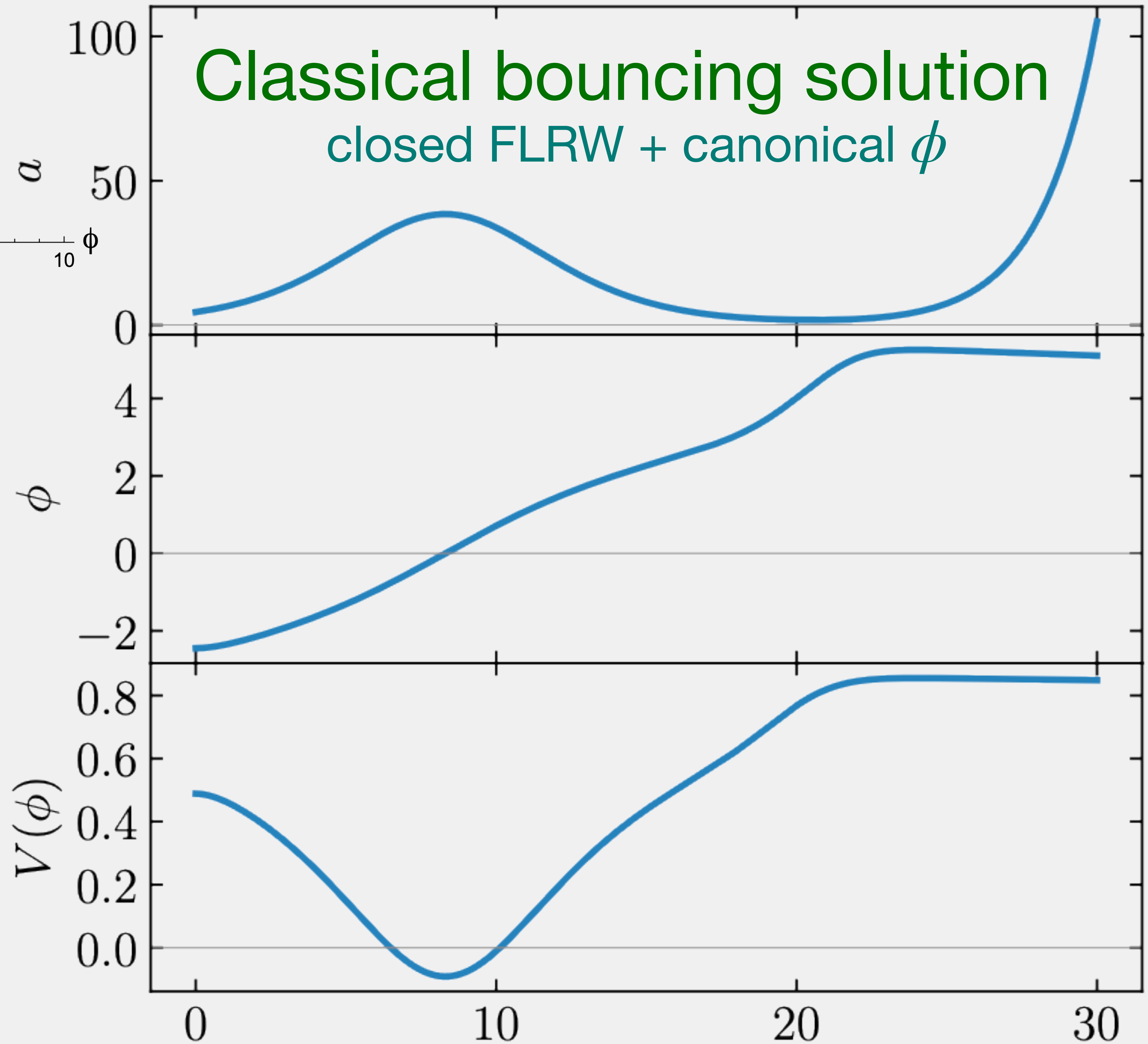
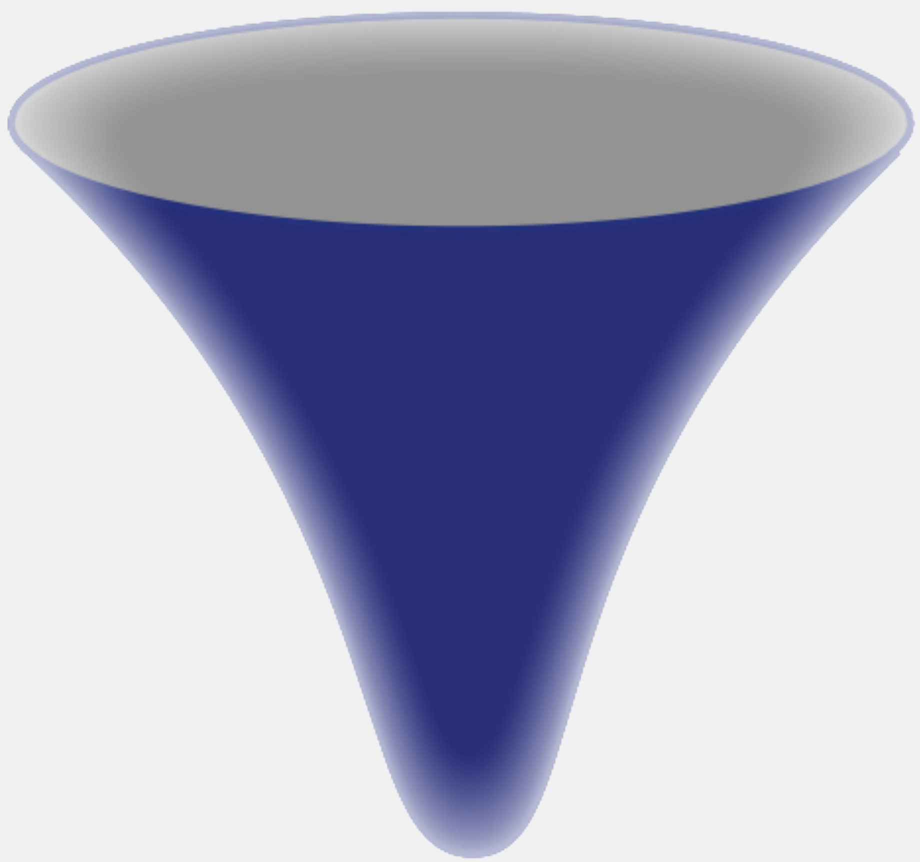


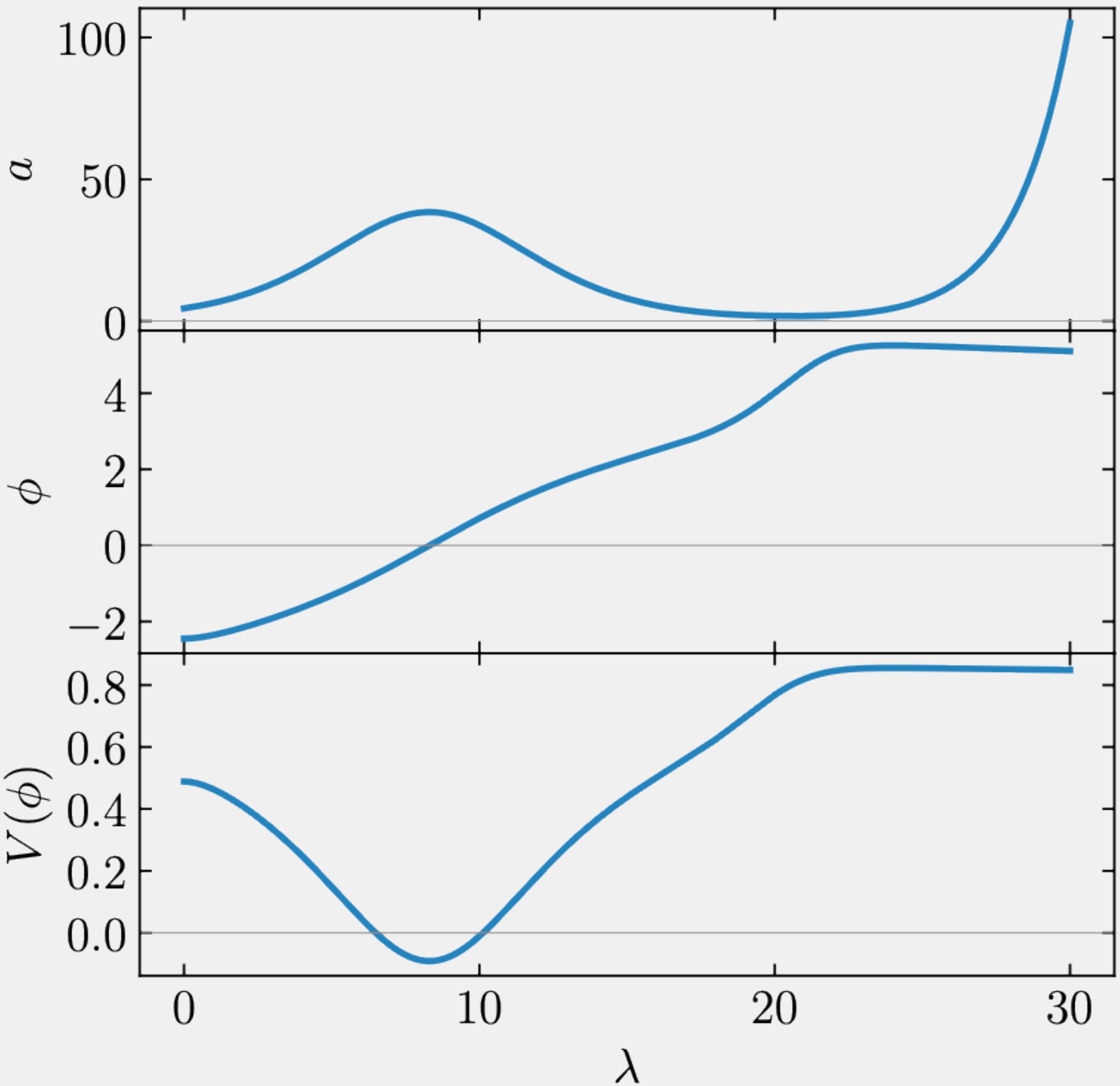


Lorentzian proper time → λ

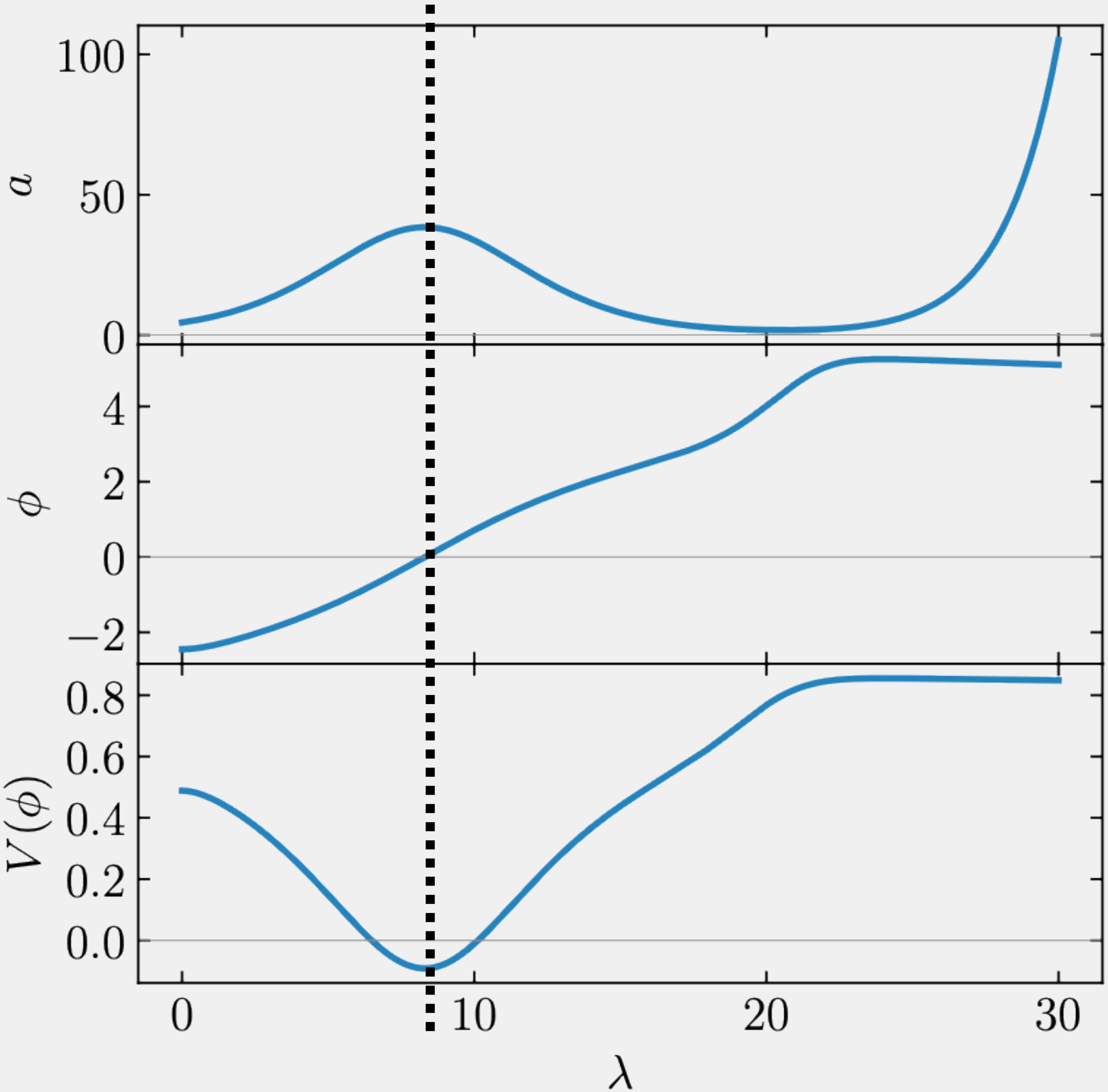


de Sitter-like instanton

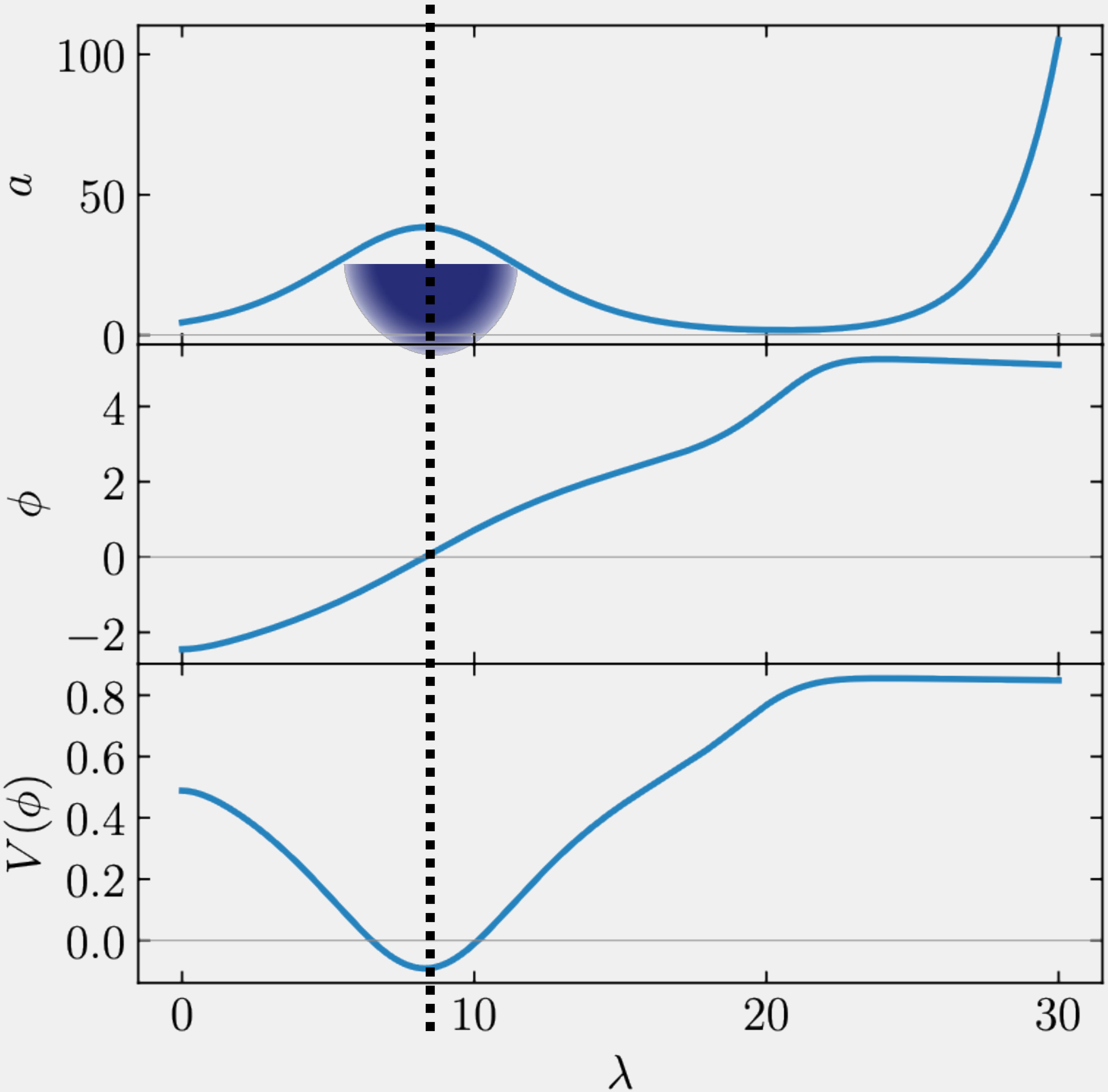
bouncing instanton



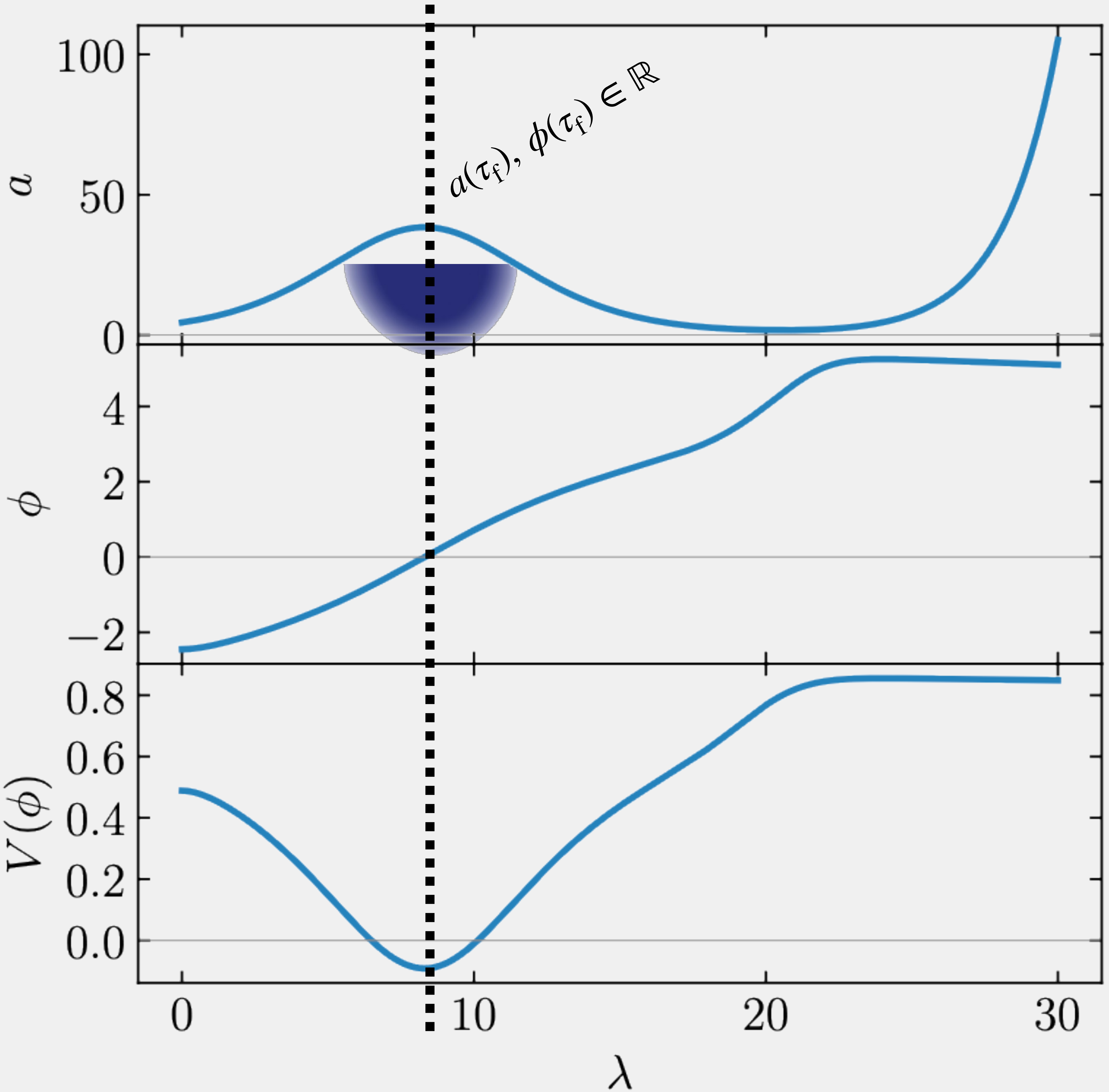
100
50
0
a
4
2
0
−2
ϕ
0.8
0.6
0.4
0.2
0.0
V(ϕ)
0
10
20
30
λ



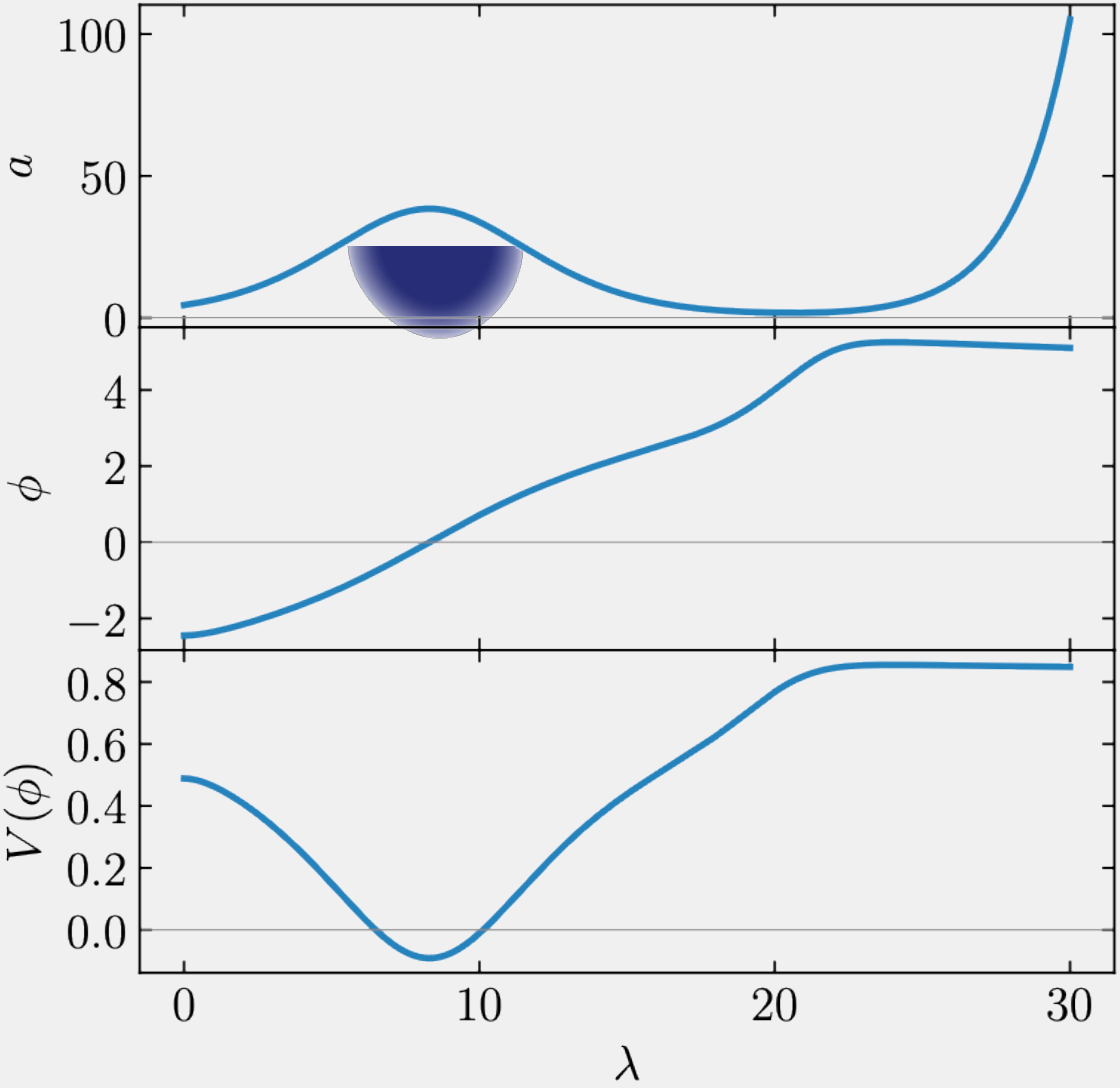
a
100
50
0
ϕ
4
2
0
−2
V(ϕ)
0.8
0.6
0.4
0.2
0.0
0
10
20
30
λ



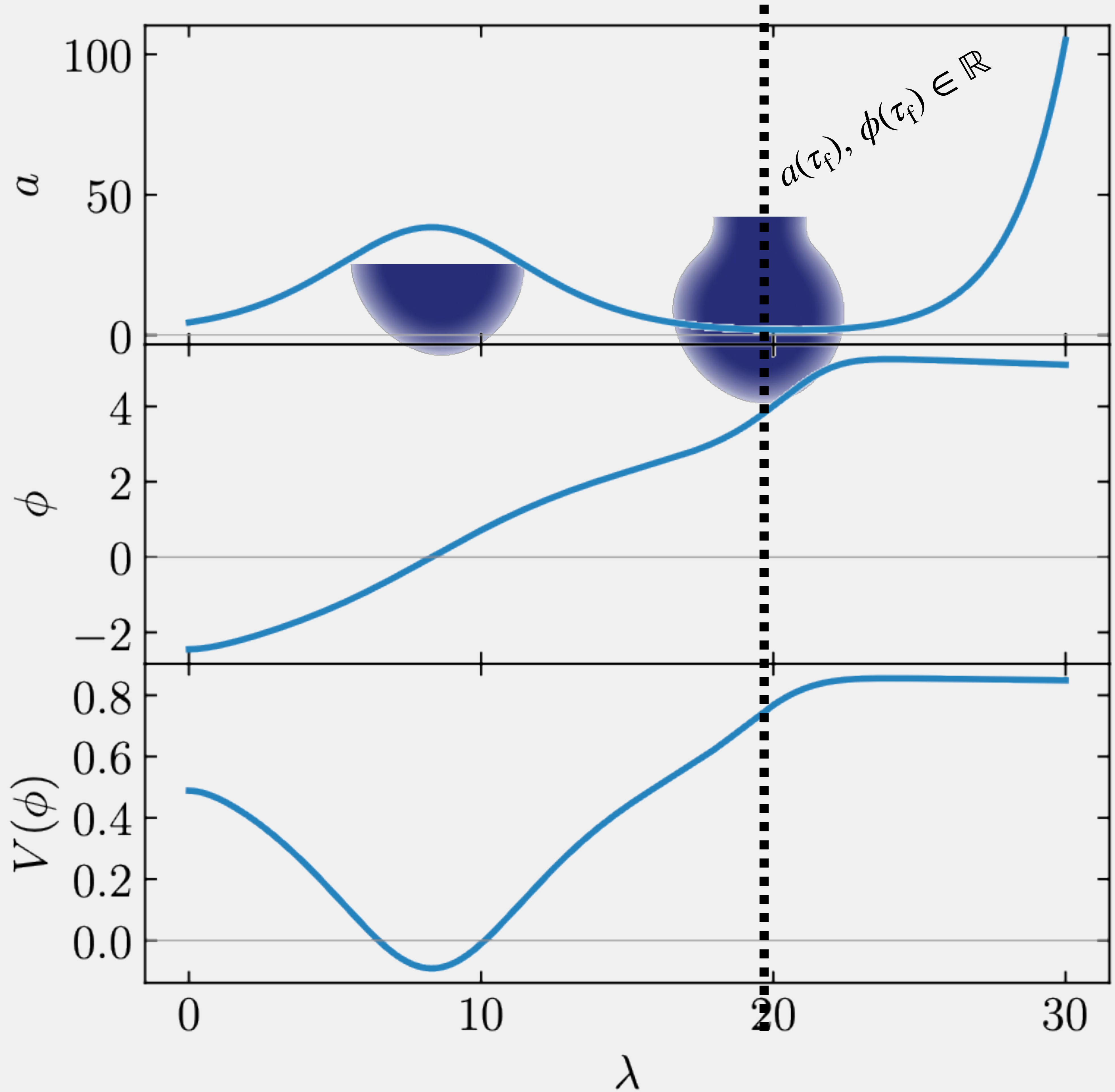
100
50
0
a
4
2
0
-2
ϕ
0.8
0.6
0.4
0.2
0.0
$V(\phi)$
0
10
20
30
λ



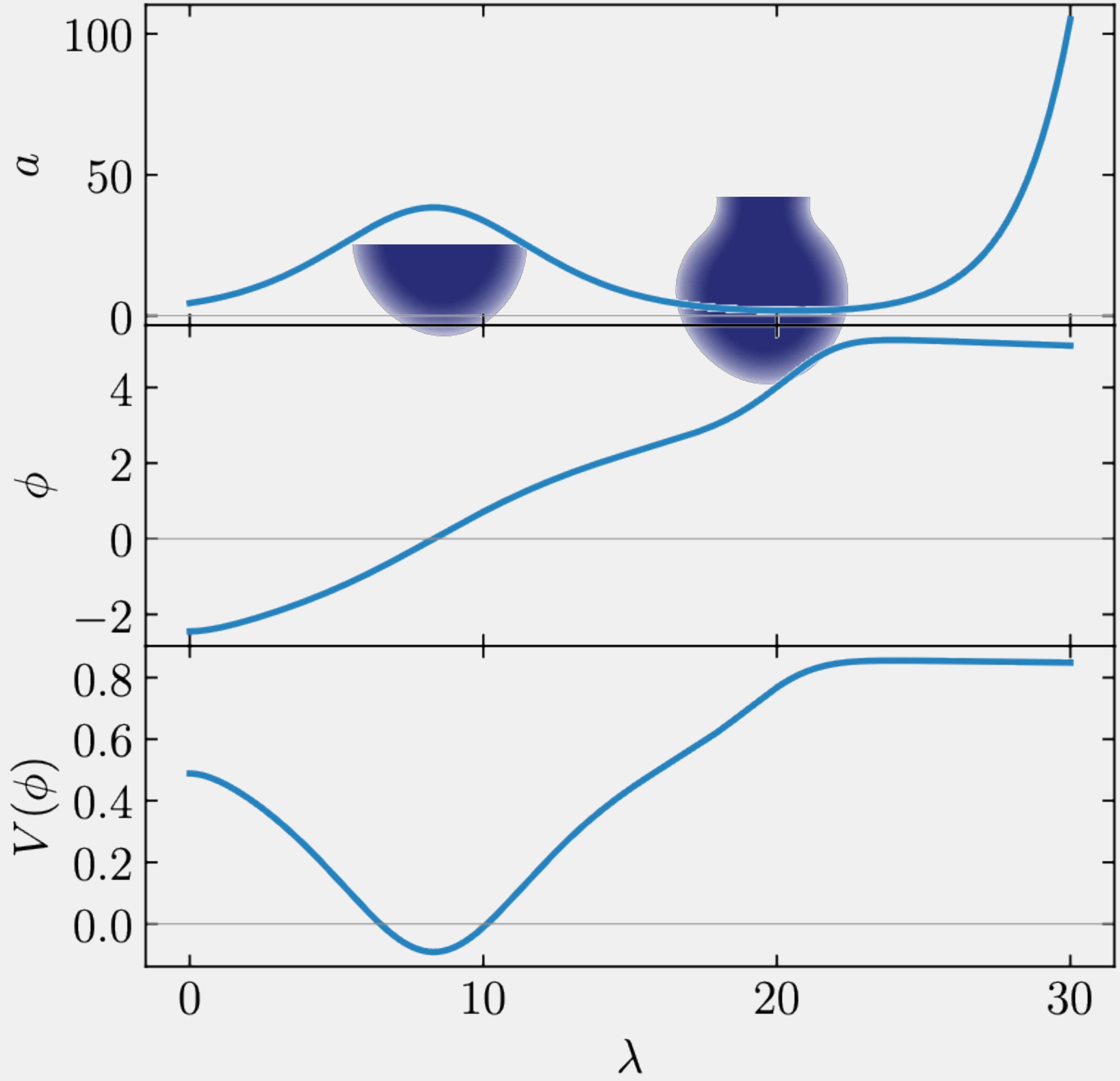
$a(\tau_f), \phi(\tau_f) \in \mathbb{R}$
a
100
50
0
ϕ
4
2
0
-2
$V(\phi)$
0.8
0.6
0.4
0.2
0.0
0
10
20
30
λ



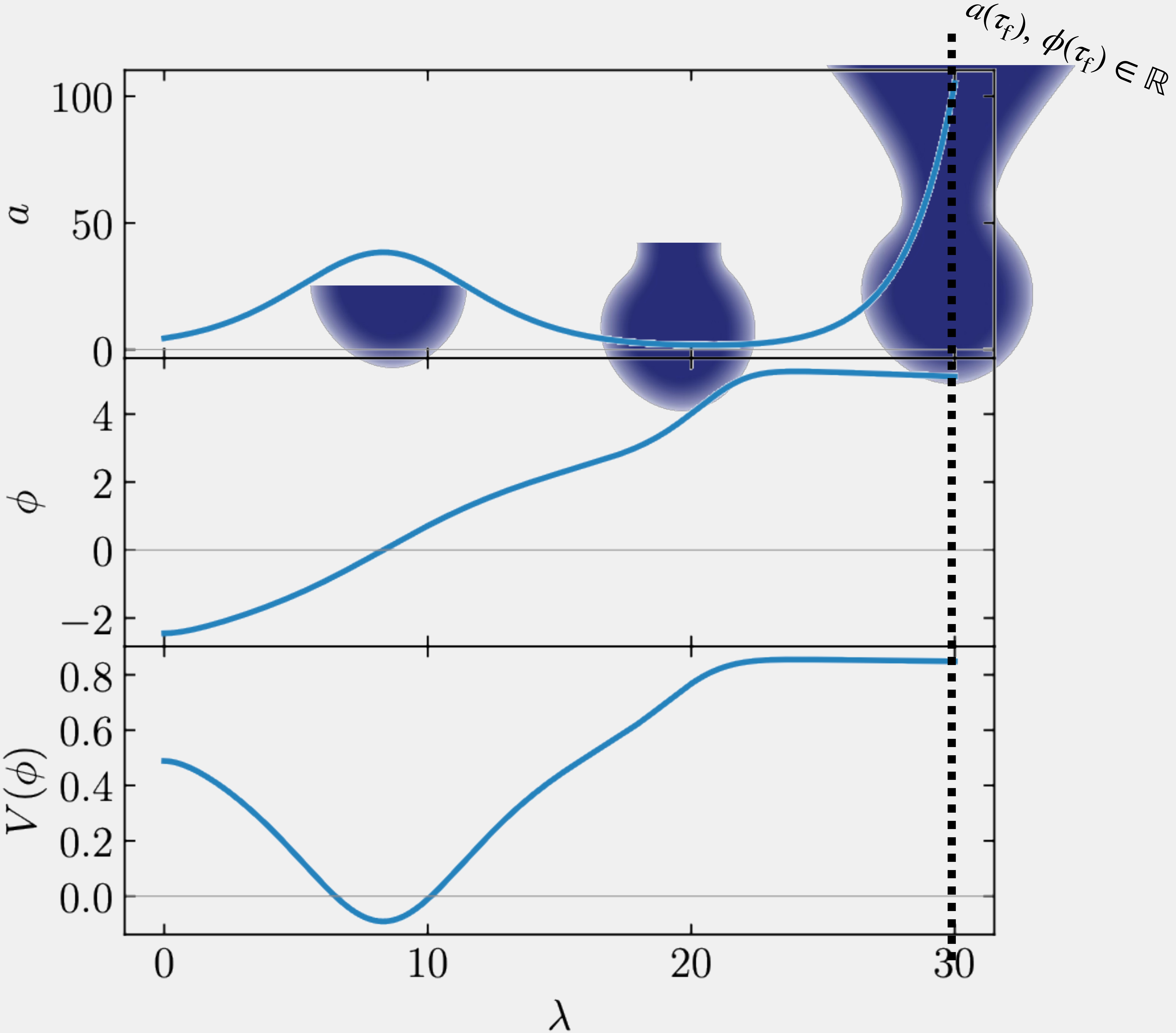
100
50
0
a
4
2
0
−2
ϕ
0.8
0.6
0.4
0.2
0.0
$V(\phi)$
0
10
20
30
λ



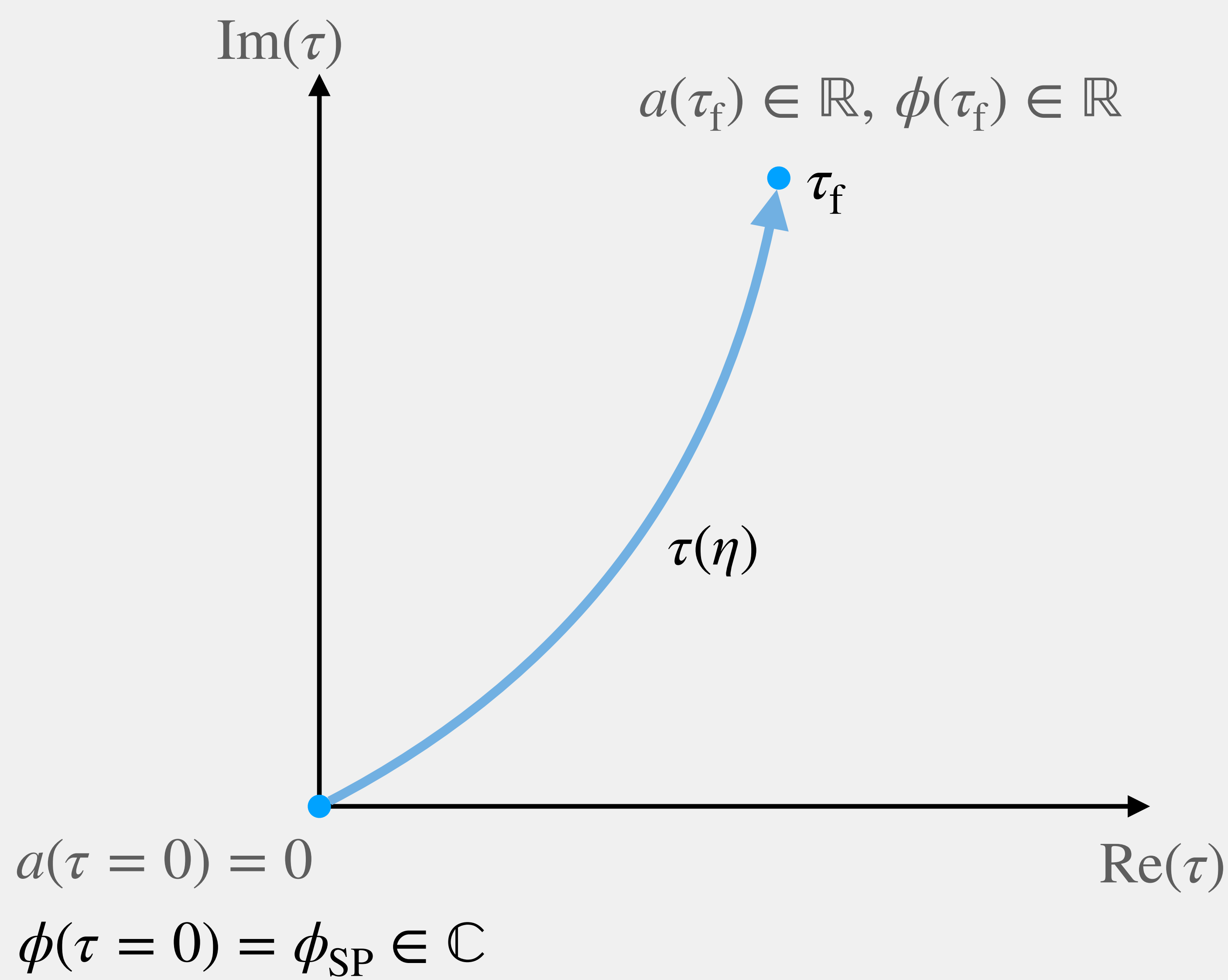
a
100
50
0
a(τ_f), ϕ(τ_f) ∈ ℝ
ϕ
4
2
0
−2
V(ϕ)
0.8
0.6
0.4
0.2
0.0
0
10
20
30
λ



100
50
0
a
4
2
0
−2
ϕ
0.8
0.6
0.4
0.2
0.0
$V(\phi)$
0
10
20
30
λ

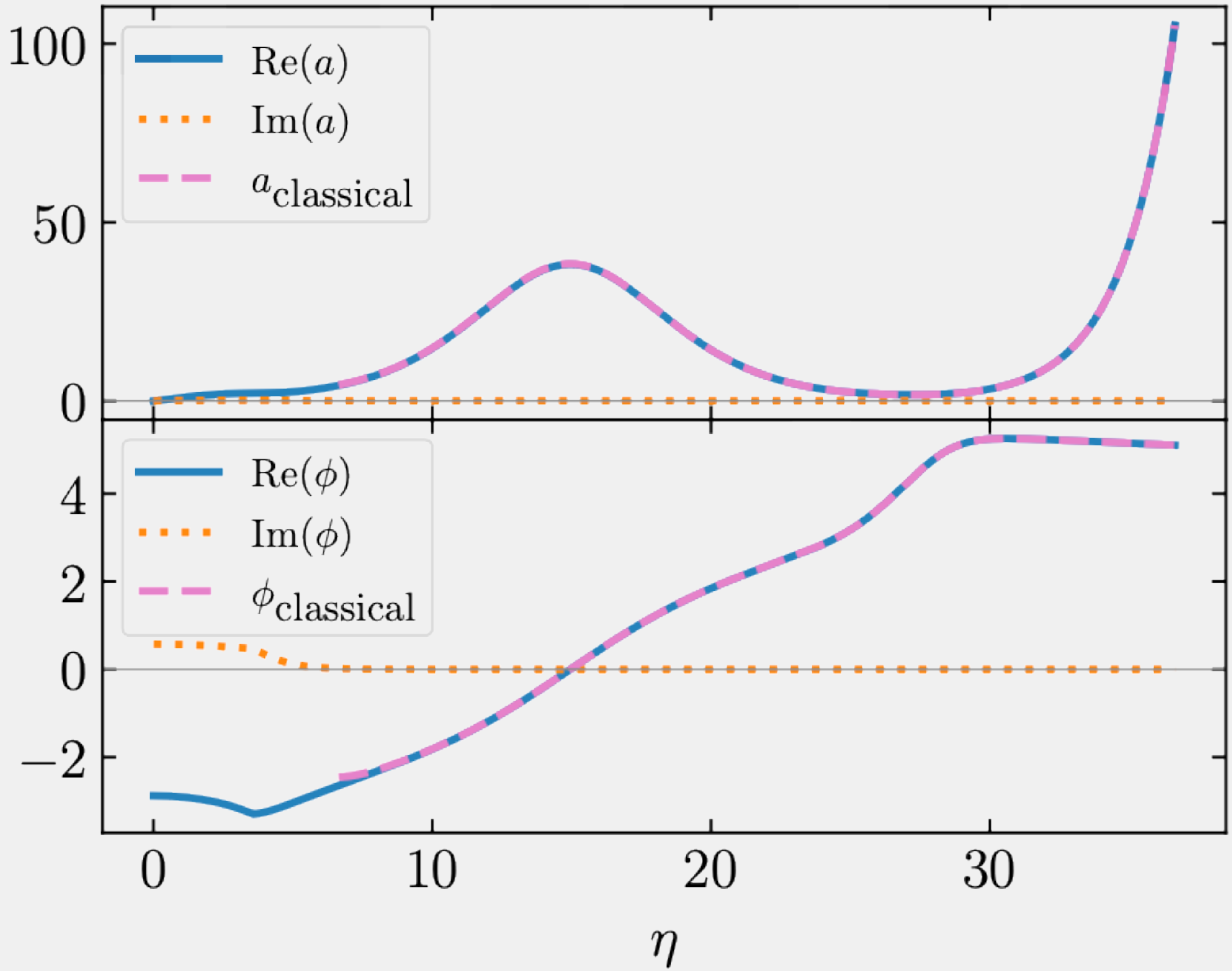

$a(\tau_f)$, $\phi(\tau_f) \in \mathbb{R}$
a
100
50
0
ϕ
4
2
0
−2
$V(\phi)$
0.8
0.6
0.4
0.2
0.0
0
10
20
30
λ

$$
\begin{cases}
a_{,\tau\tau} + \dfrac{a}{3}\left((\phi_{,\tau})^2 + V\right) = 0 \\[2ex]
\phi_{,\tau\tau} + 3\dfrac{a_{,\tau}}{a}\phi_{,\tau} - V_{,\phi} = 0 \\[2ex]
(a_{,\tau})^2 - 1 = \dfrac{a^2}{3}\left(\dfrac{1}{2}(\phi_{,\tau})^2 - V\right)
\end{cases}
$$

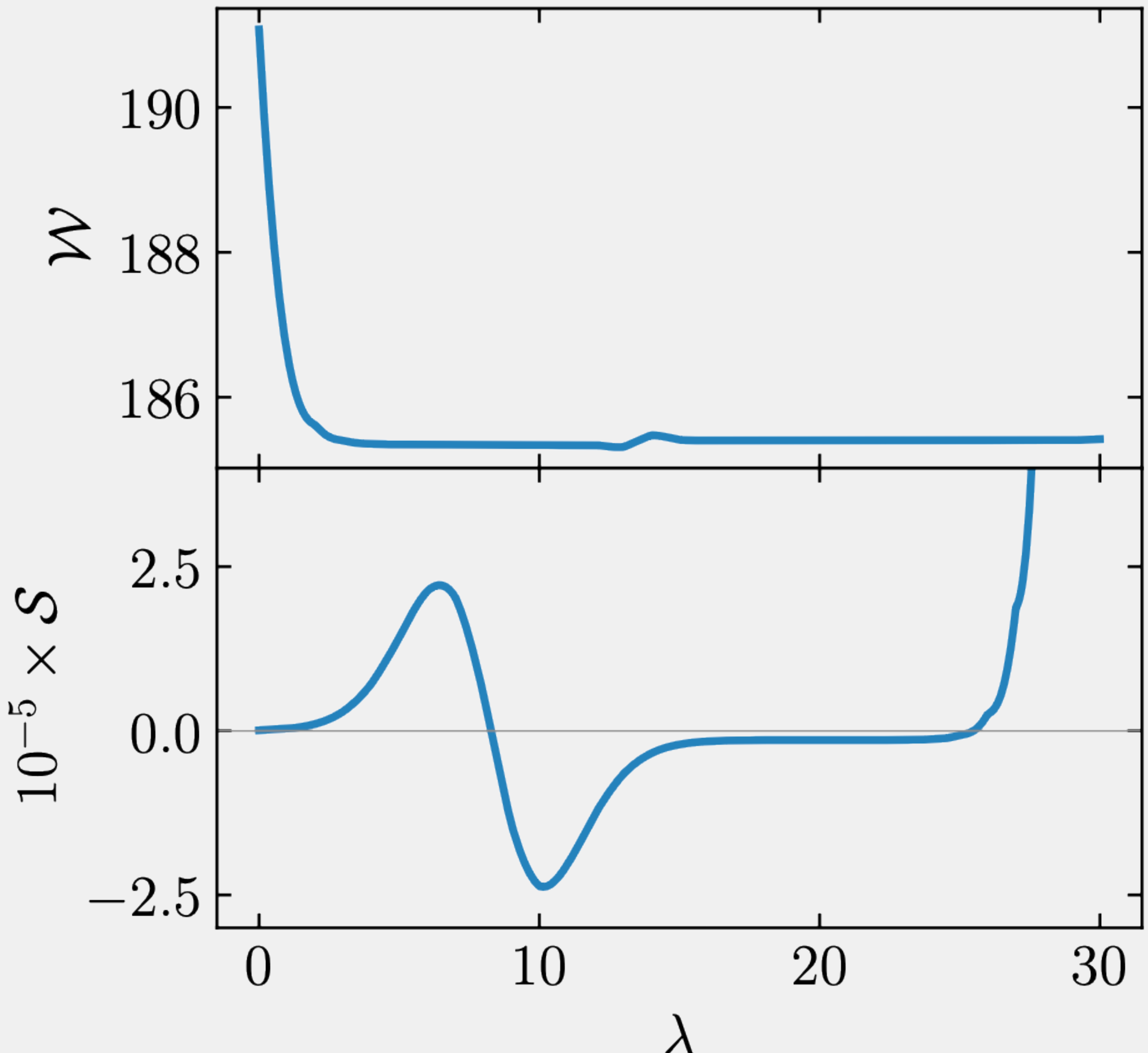


Lehners-JQ [JCAP01(2025)027]

Bouncing instanton solution at $\lambda = 30$



Wave function $\Psi \sim \exp\left(\frac{1}{\hbar}(\mathscr{W} + i\mathscr{S})\right)$



Lehners-JQ [JCAP01(2025)027]

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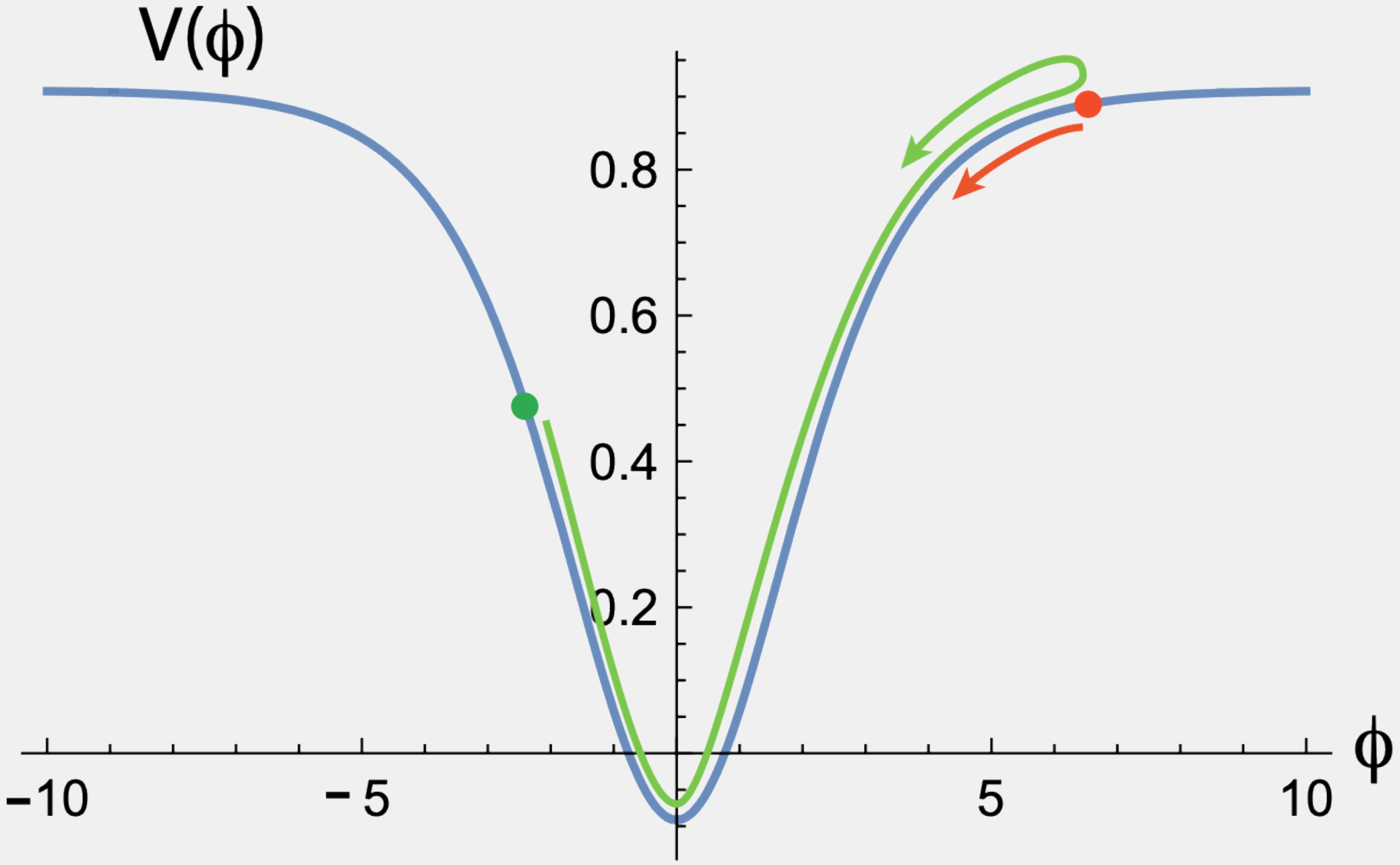
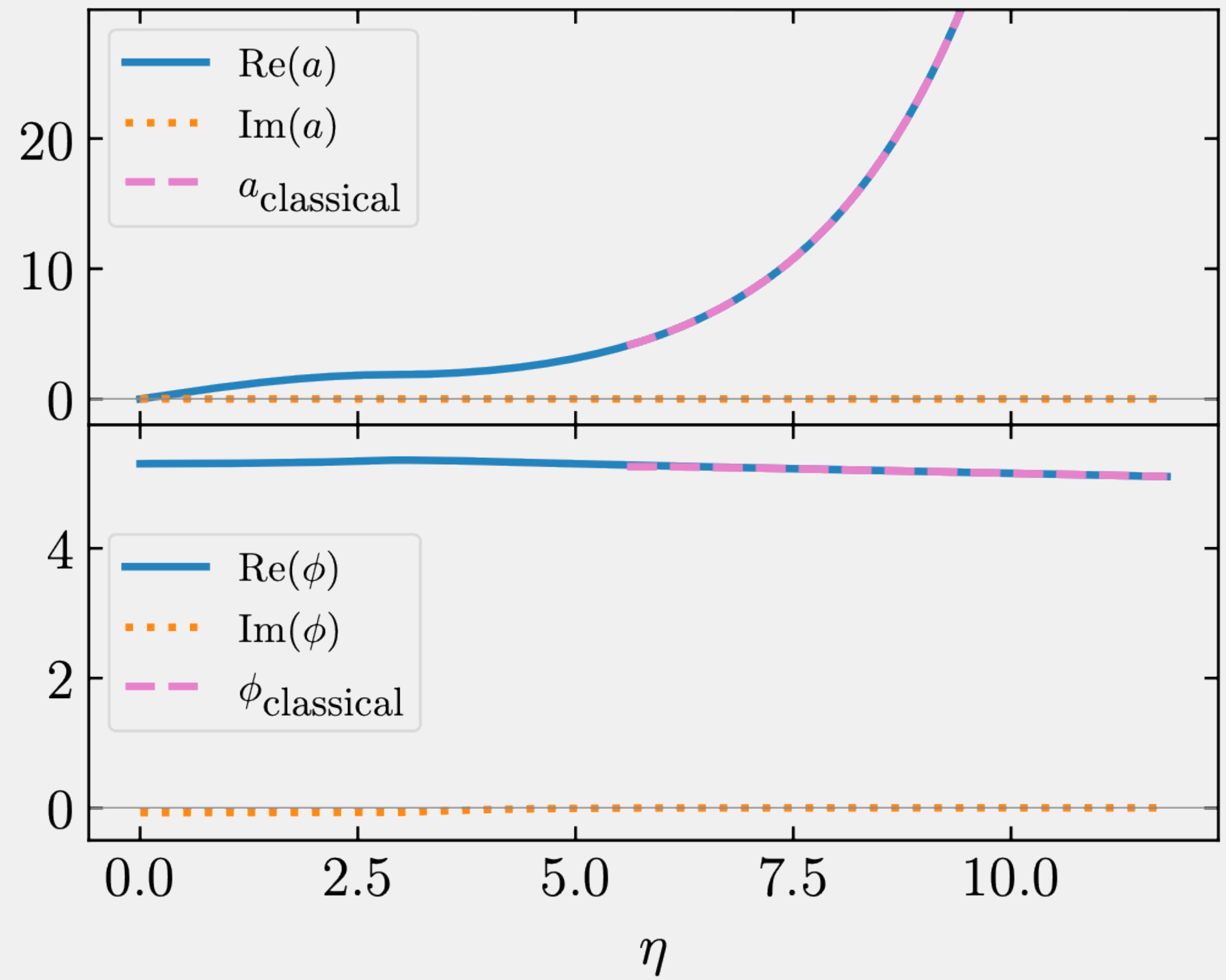
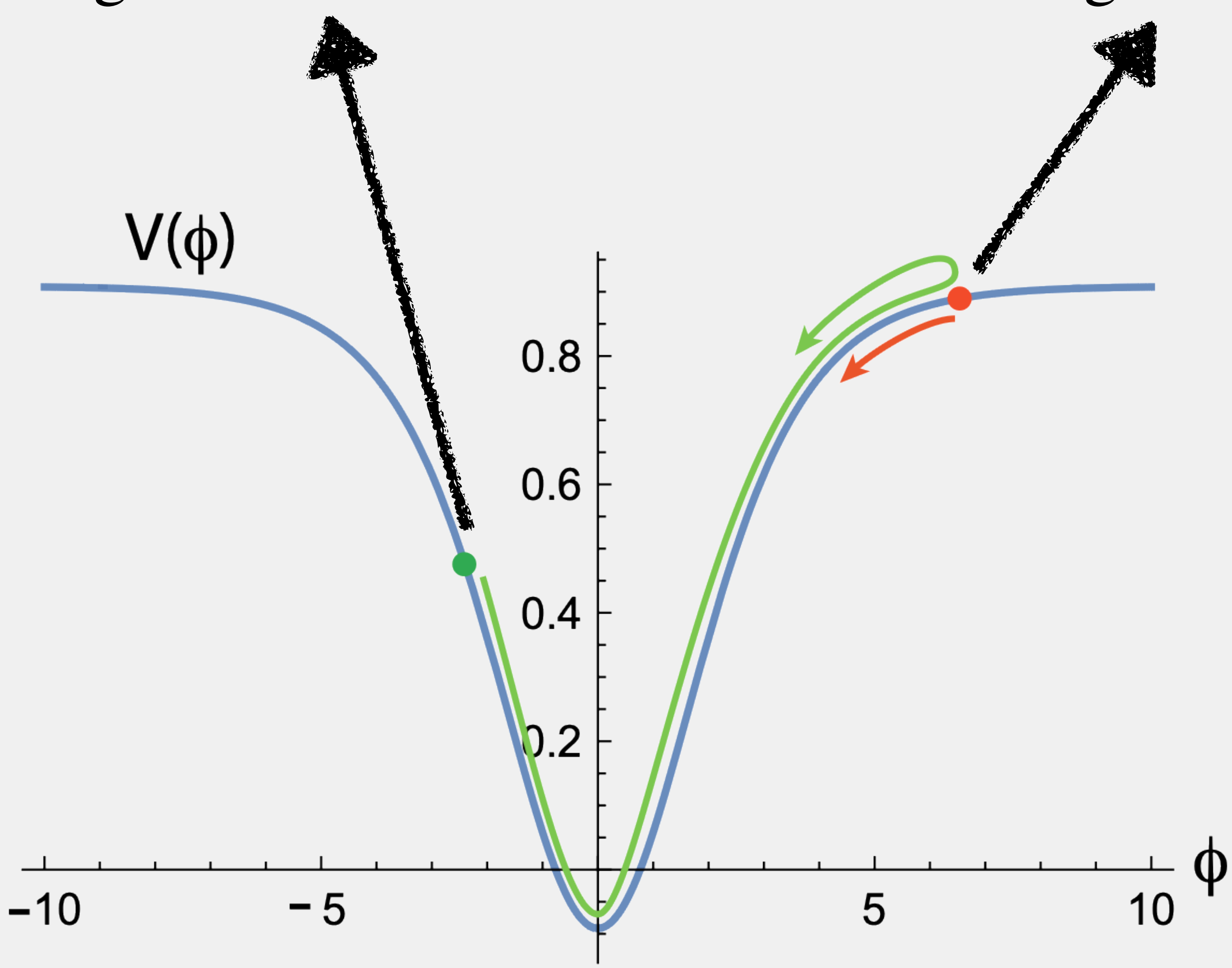


Inflationary instanton solution
for the same final conditions



$$|\Psi| \sim \exp\left(\frac{\mathscr{W}}{\hbar}\right) \approx \exp\left(\frac{12\pi^2}{\hbar V\big(\mathrm{Re}(\phi_{\mathrm{SP}})\big)}\right)$$

$$\mathscr{W}_{\text{bouncing}} \approx 185.4\,, \quad \mathscr{W}_{\text{non-bouncing}} \approx 137.9$$



Takeaways so far

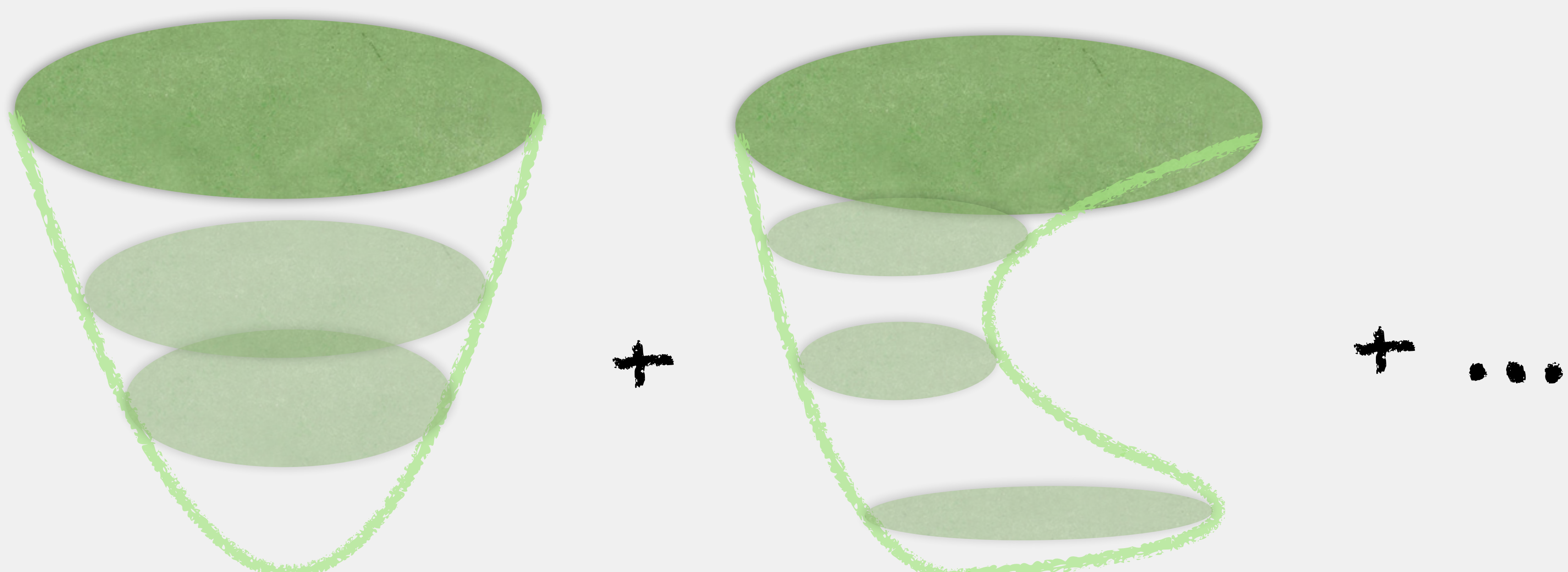
- Bounce has a higher weighting, but has more fine-tuned initial conditions…

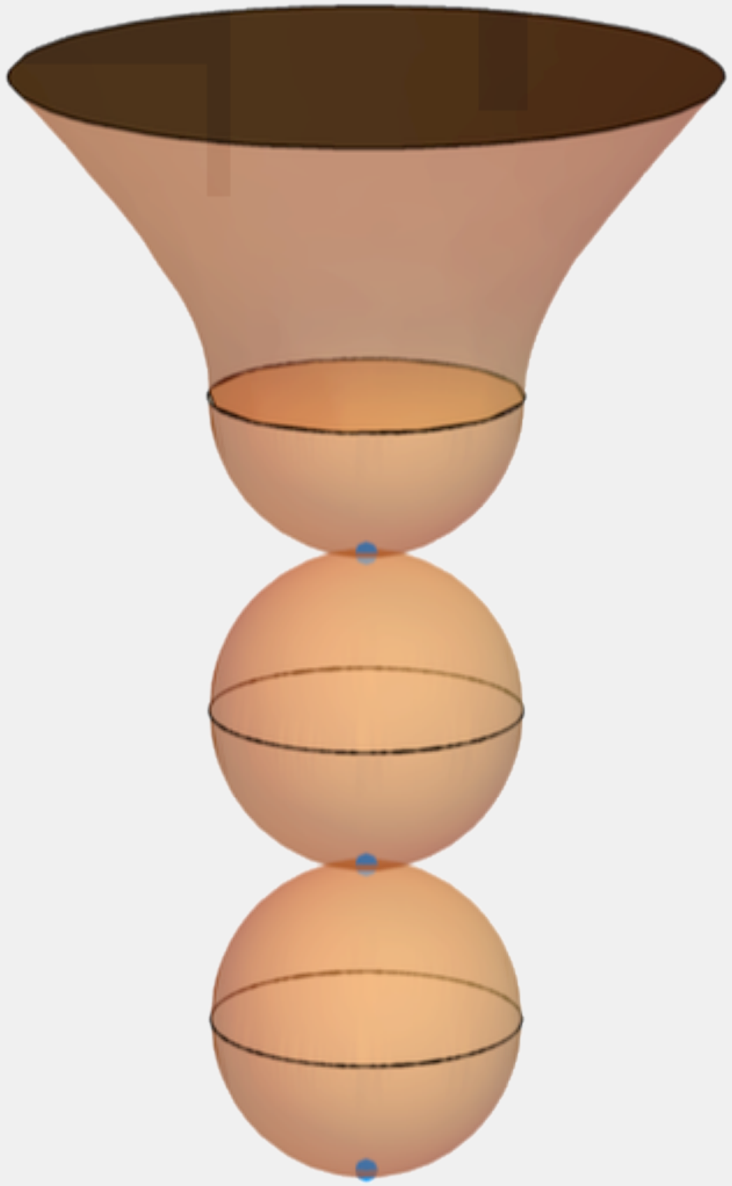
  → not quite conclusive yet

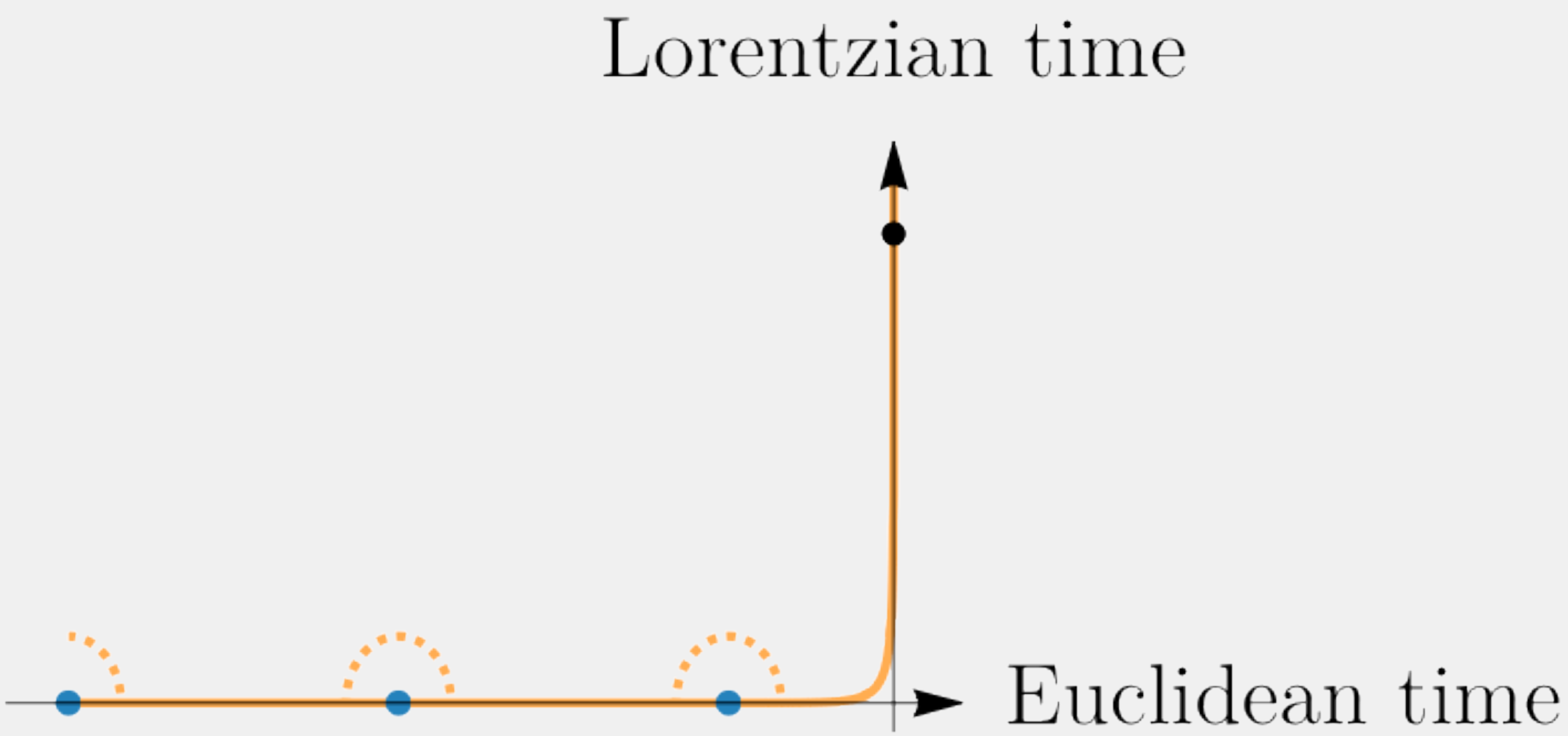
- Pre-inflationary bouncing instantons that emerge into a classical universe exist!

Part II

Should we really include all complex geometries in the path integral?







- Adding n spheres enhances the wave function by powers of n:

$$\Psi \sim \exp\left[\left(n+\frac{1}{2}\right)\frac{\left|S_{\mathrm{E}}[S^4]\right|}{\hbar}\right]$$

Kontsevich-Segal criterion

[2105.10161]

- Consider an arbitrary real non-zero p-form gauge field A with associated q-form field strength $F = \mathrm{d}A$, $q = p + 1$, on a fixed background g:

$$S_{\mathrm{E}}[g, A] = \int F \wedge \star F = \frac{1}{2q!} \int \mathrm{d}^D x \sqrt{\det g_{\alpha\beta}}\, g^{\mu_1\nu_1} \cdots g^{\mu_q\nu_q} F_{\mu_1\cdots\mu_q} F_{\nu_1\cdots\nu_q}$$

- Then $\int \mathscr{D}A\, e^{-S_{\mathrm{E}}[g,A]/\hbar}$ converges if $\mathrm{Re}(S_{\mathrm{E}}[g, A]) > 0$

- If we diagonalise the metric as $g_{\mu\nu} = \lambda_{(\mu)}\delta_{\mu\nu}$, then $\sqrt{\det g_{\alpha\beta}} = \prod_{\mu=0}^{D-1} \sqrt{\lambda_{(\mu)}}$

and the convergence condition becomes

$$\mathrm{Re}\left(\prod_{\mu=0}^{D-1} \sqrt{\lambda_{(\mu)}} \prod_{\mu \in S} \lambda_{(\mu)}^{-1} \right) > 0 \;\forall\, S \subseteq \{0, \ldots, D-1\} \quad \Longleftrightarrow \quad \Sigma \equiv \sum_{\mu=0}^{D-1} \left| \mathrm{Arg}(\lambda_{(\mu)}) \right| < \pi$$

real Lorentzian -+++ $\Rightarrow$ $\mathrm{Arg}(\lambda_{(0)}) = \pi$, $\mathrm{Arg}(\lambda_{(i)}) = 0$ $\Rightarrow$ $\Sigma = \pi$

saturated bound

↓

conditionally convergent path integral

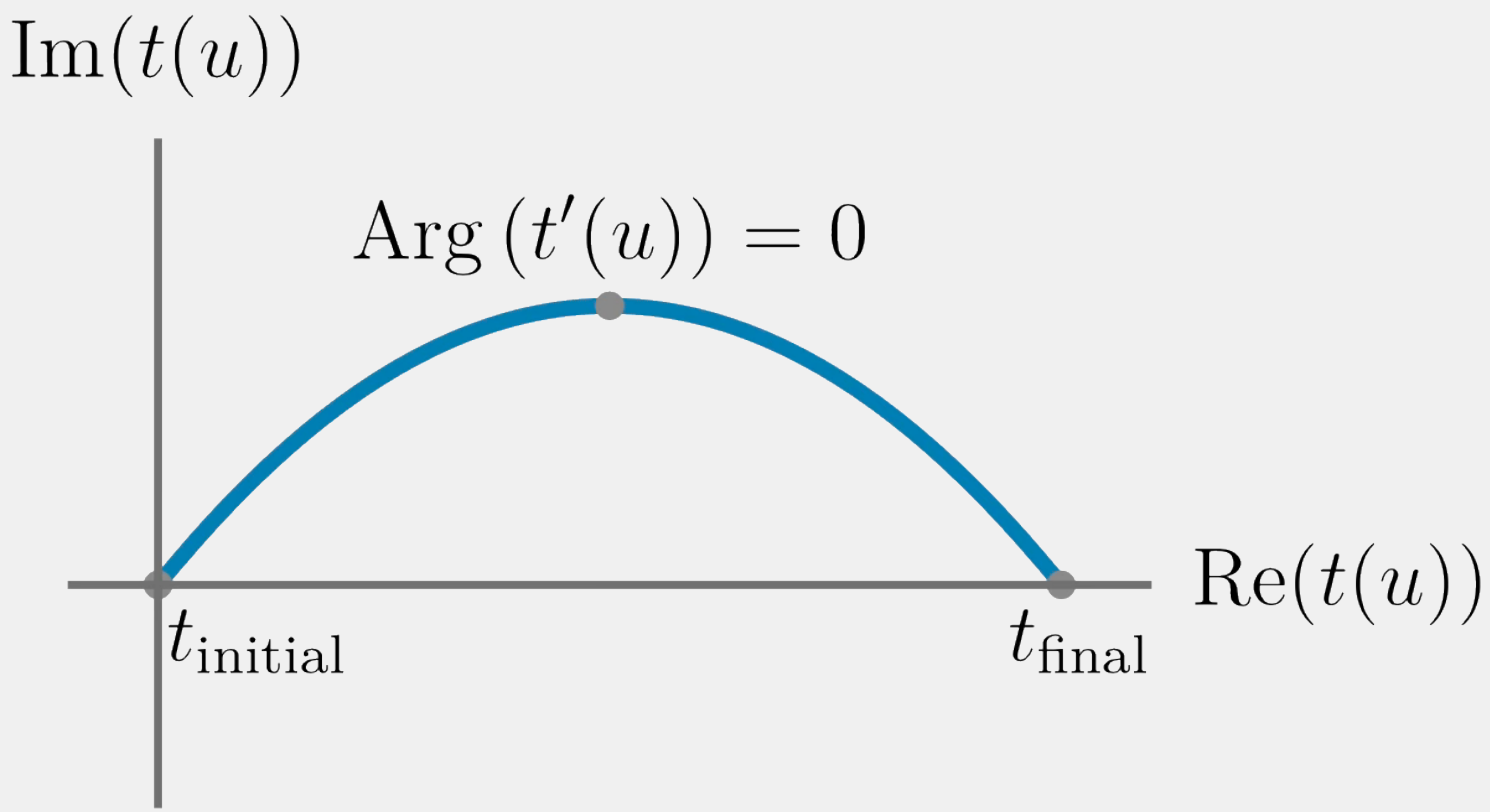
↓ regulate

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(1 \mp i\varepsilon)\mathrm{d}t^2 + h_{ij}\mathrm{d}x^i \mathrm{d}x^j$$

Witten [2111.06514], Visser [2111.14016]

True for any complex path

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a(t)^2\mathrm{d}\vec{x}^2 = -\,t'(u)^2\mathrm{d}u^2 + a(t(u))^2\mathrm{d}\vec{x}^2$$



$$\Rightarrow \Sigma(u) = \left|\mathrm{Arg}\left(-t'(u)^2\right)\right| + 3\left|\mathrm{Arg}\left(a(t(u))^2\right)\right| \geq \left|\mathrm{Arg}(-t'(u)^2)\right| \overset{!}{=} \pi$$

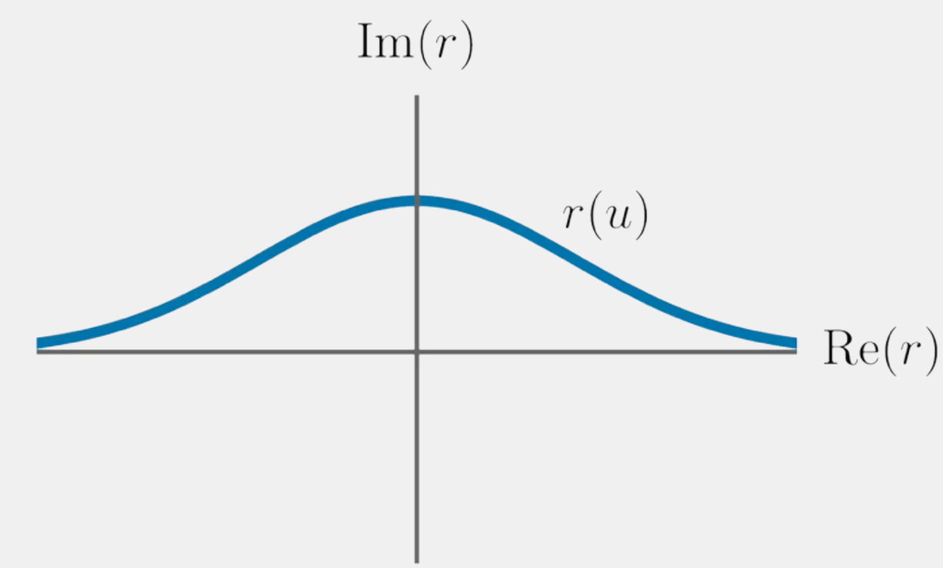
Witten [2111.06514]

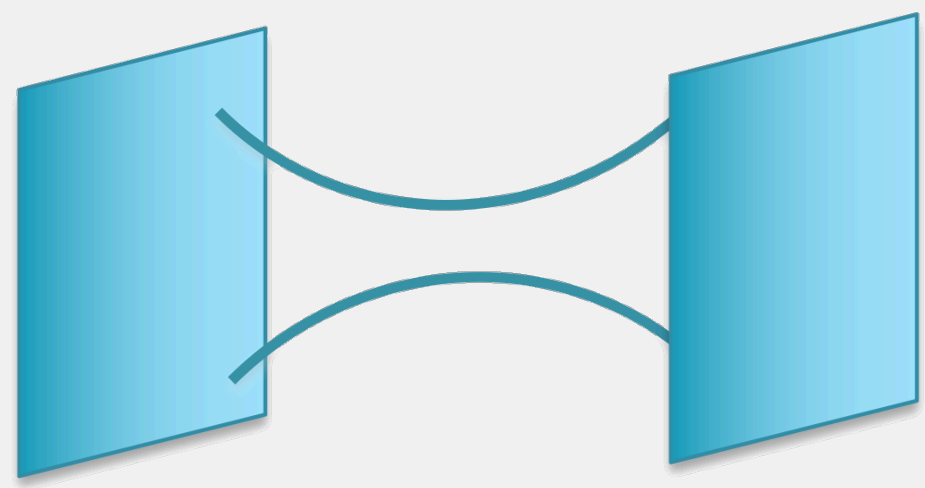
$$\int \mathscr{D}g \int \mathscr{D}A \, e^{-S_{\mathrm{E}}[g,A]/\hbar}$$

Example: (3D)

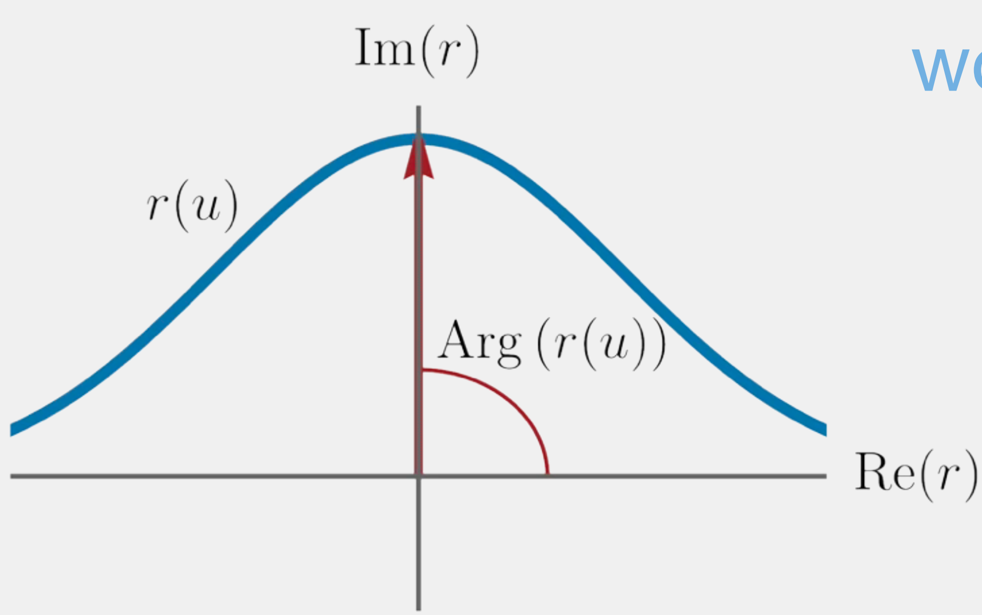
$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{(2)} \qquad r \mapsto r(u) \in \mathbb{C} \,, \; u \in \mathbb{R}$$

$$\longrightarrow \quad \mathrm{d}s^2 = r'(u)^2 \mathrm{d}u^2 + r(u)^2 \mathrm{d}\Omega^2_{(2)}$$





wormhole same weighting as flat space!?

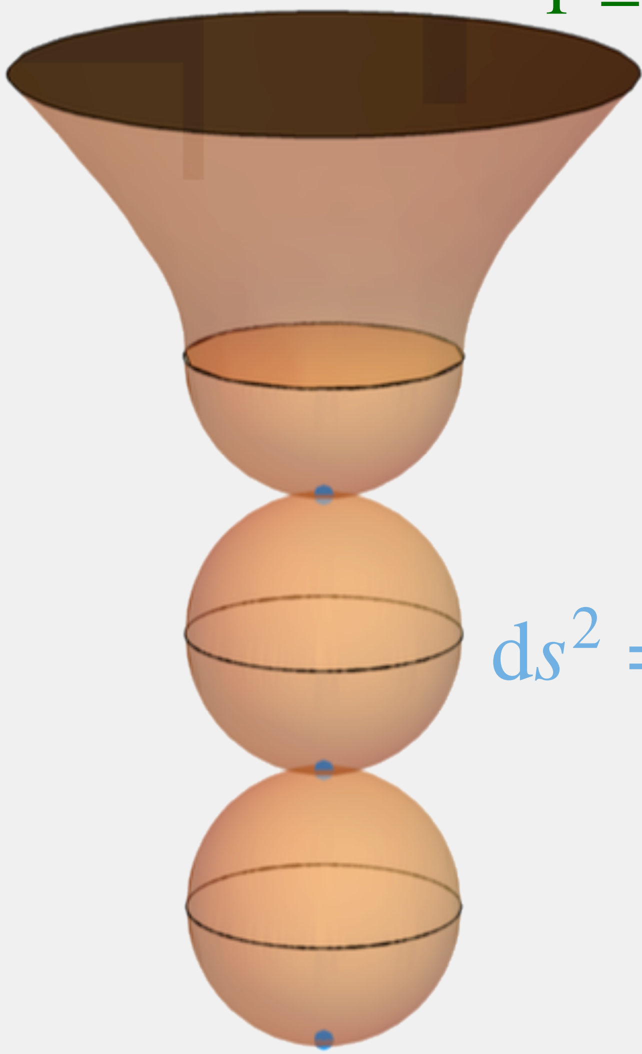


$$\mathrm{Arg}(r(u)^2) \overset{!}{=} \pi \implies \Sigma(u) \overset{!}{\geq} 2\pi \nless \pi$$

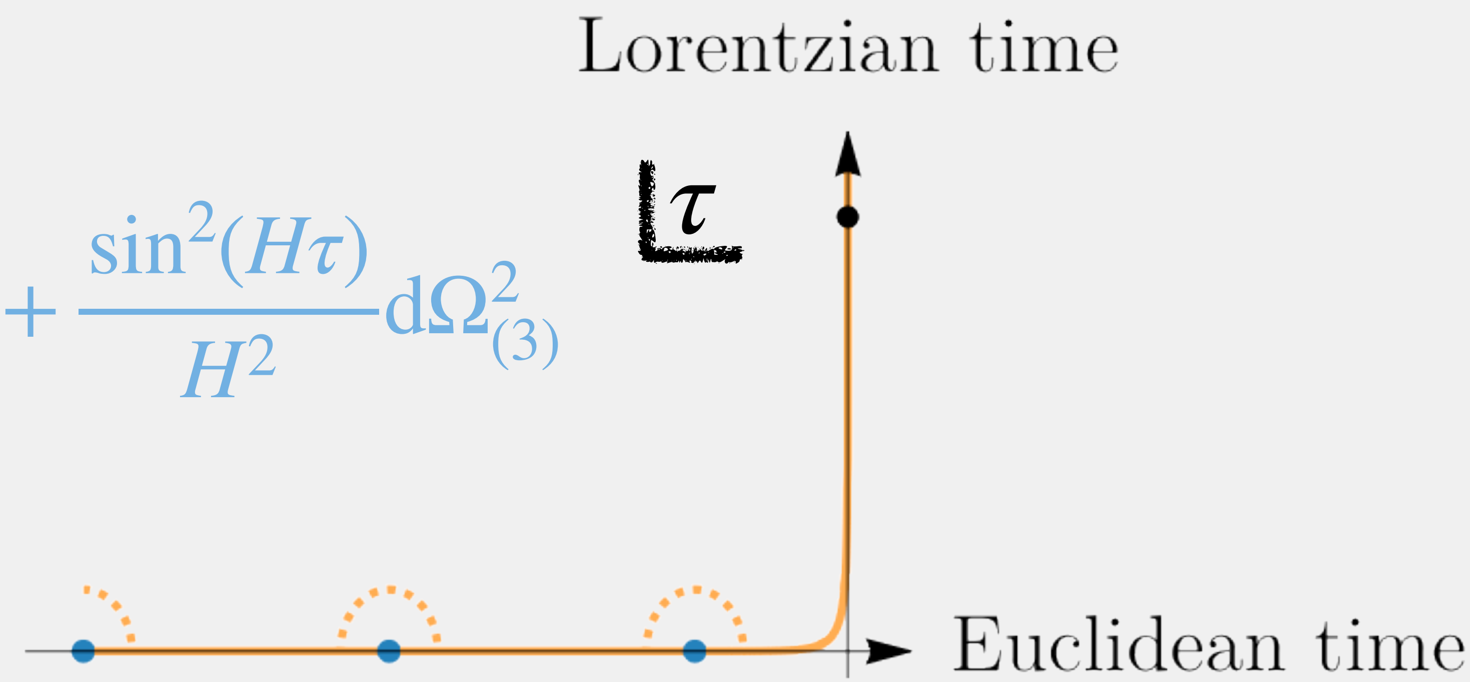
wormhole non-allowable!

Jonas-Lehners-JQ [JHEP08(2022)284]

$$\Psi = \int_{\text{no bd}} \mathscr{D}g\, e^{-S_{\mathrm{E}}[g]/\hbar}$$

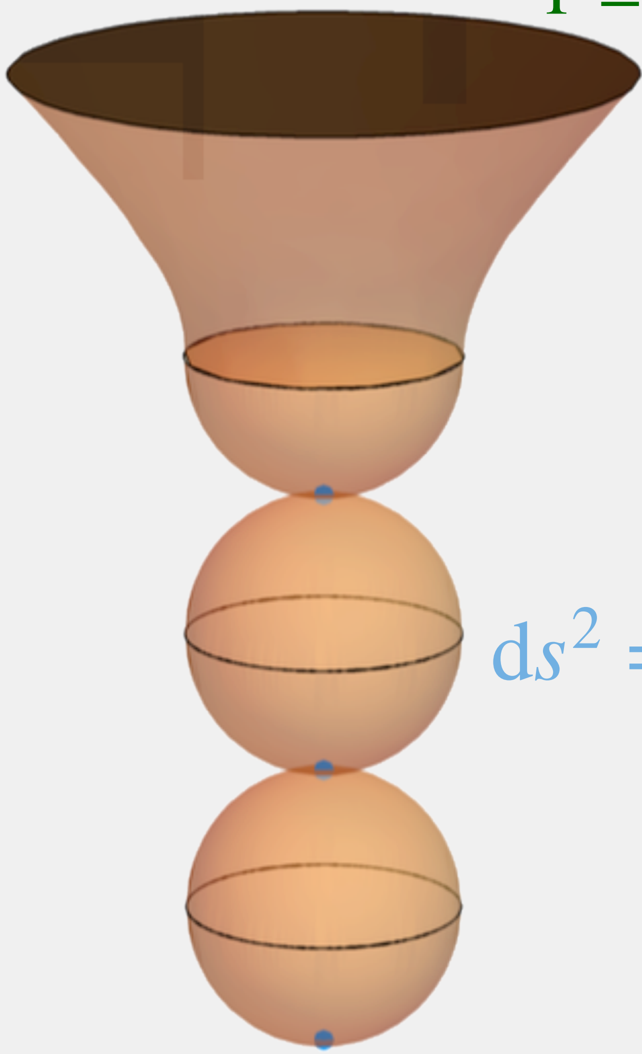


$$ds^2 = d\tau^2 + \frac{\sin^2(H\tau)}{H^2} d\Omega_{(3)}^2$$



Jonas-Lehners-JQ [JHEP08(2022)284]

$$\Psi = \int_{\text{no bd}} \mathscr{D}g\, e^{-S_E[g]/\hbar}$$



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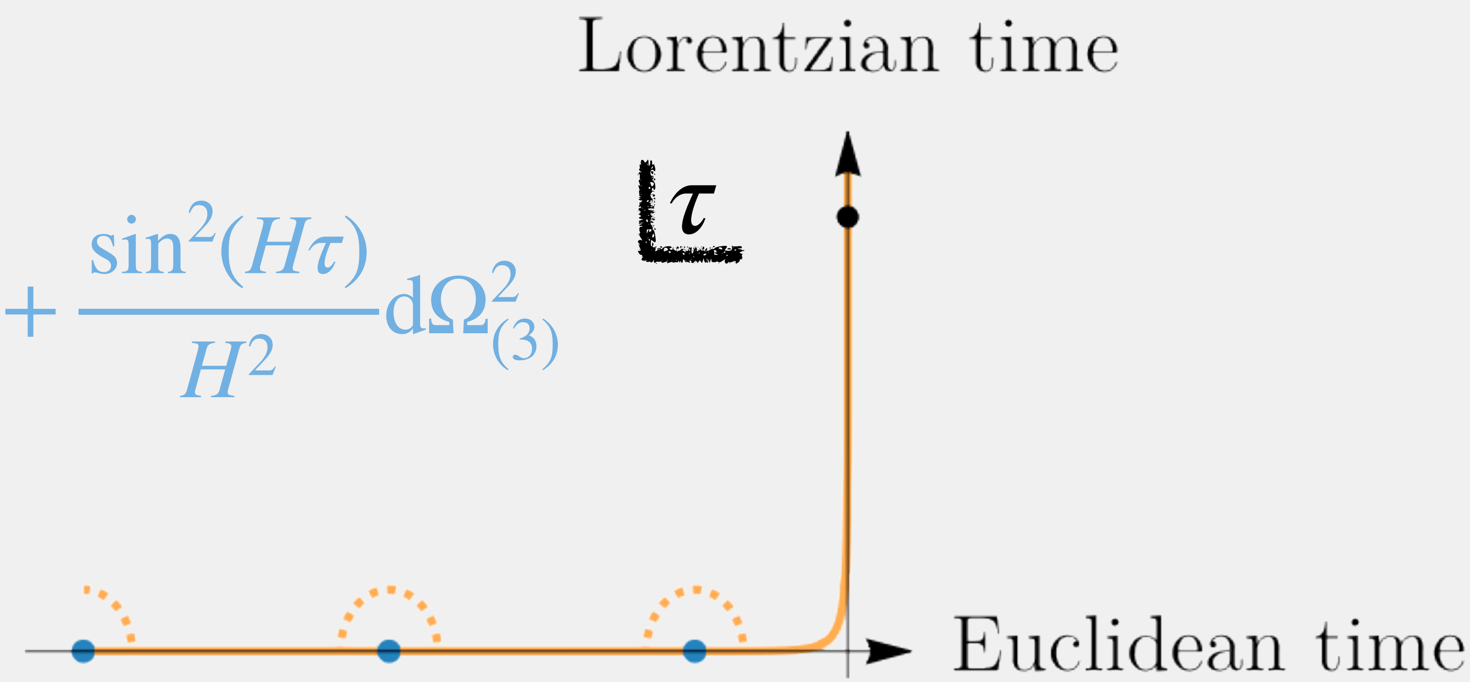

$$ds^2 = \tau'(u)^2 du^2 + a^2(\tau(u)) d\Omega_{(3)}^2\,, \quad a(\tau) = \frac{\sin(H\tau)}{H}$$

Jonas-Lehners-JQ [JHEP08(2022)284]

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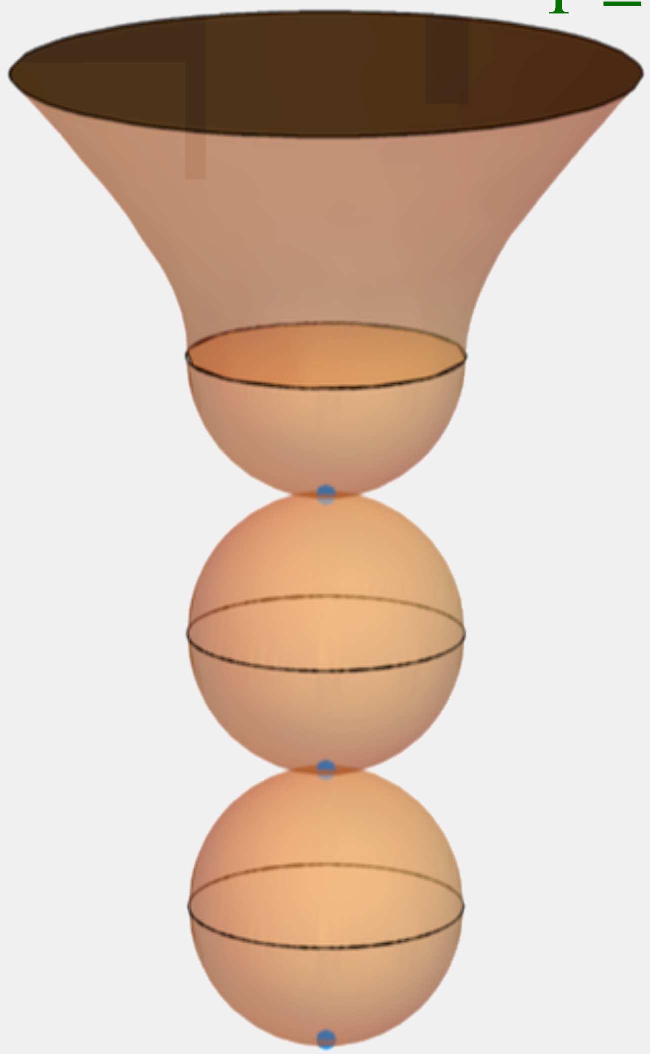

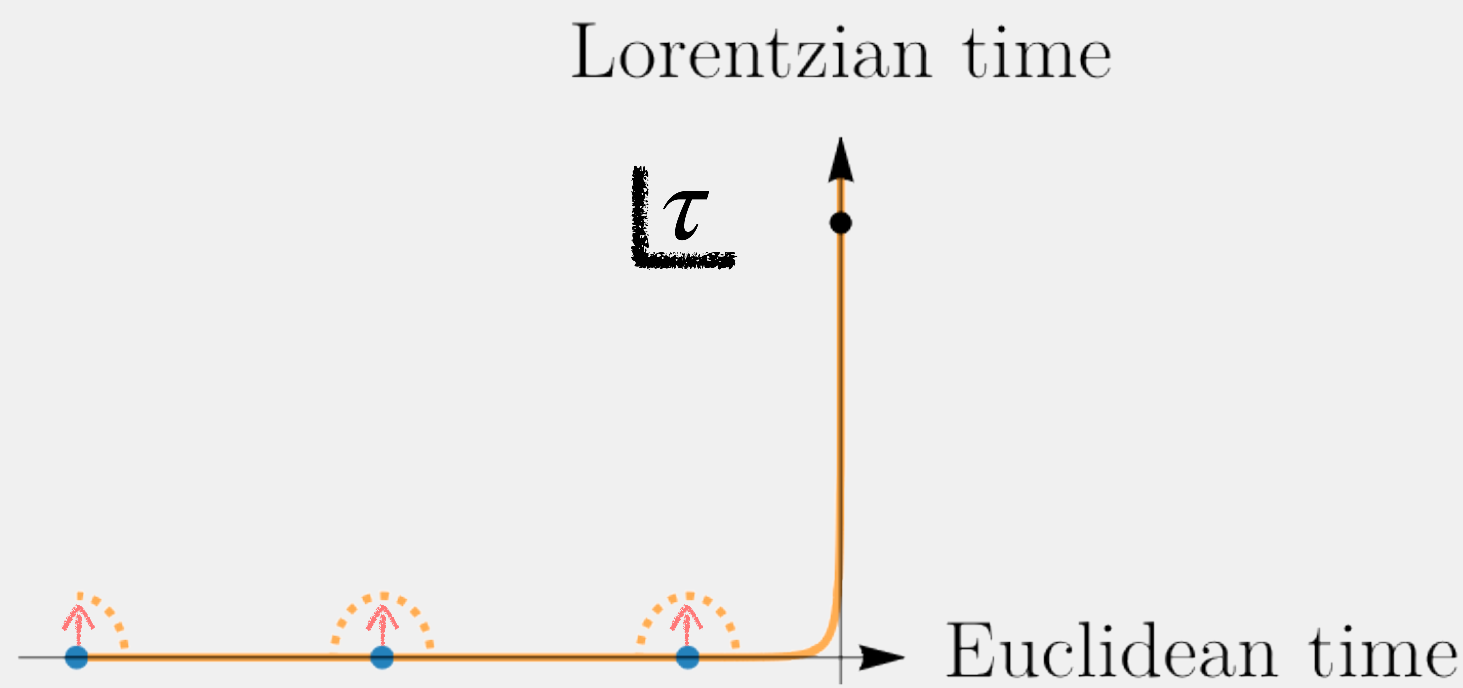

$$ds^2 = \tau'(u)^2 du^2 + a^2(\tau(u)) d\Omega^2_{(3)}, \quad a(\tau) = \frac{\sin(H\tau)}{H}$$

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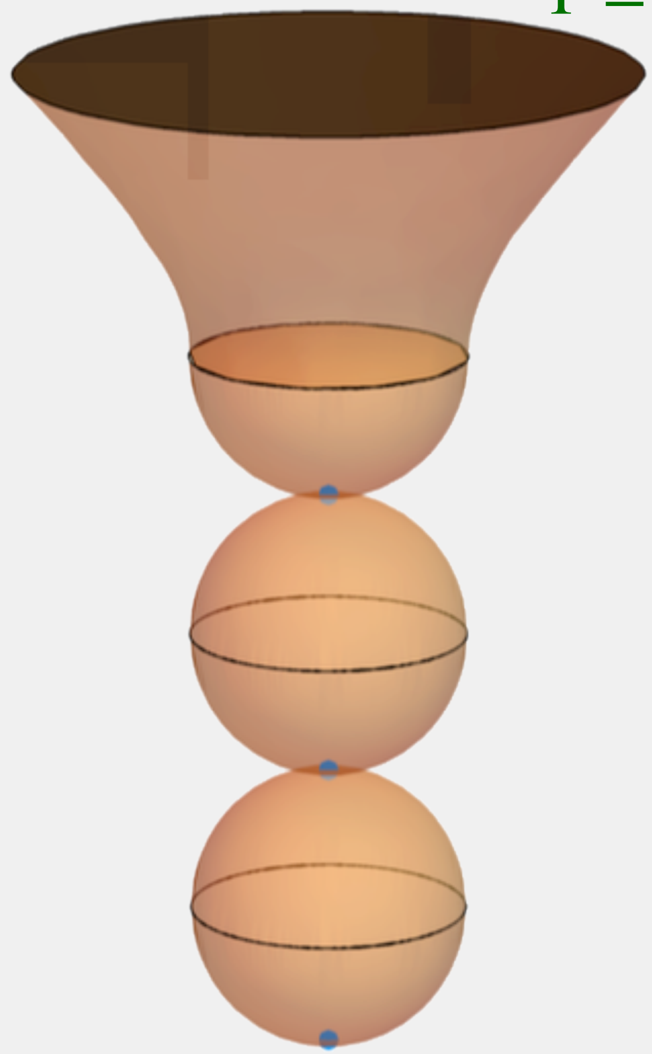

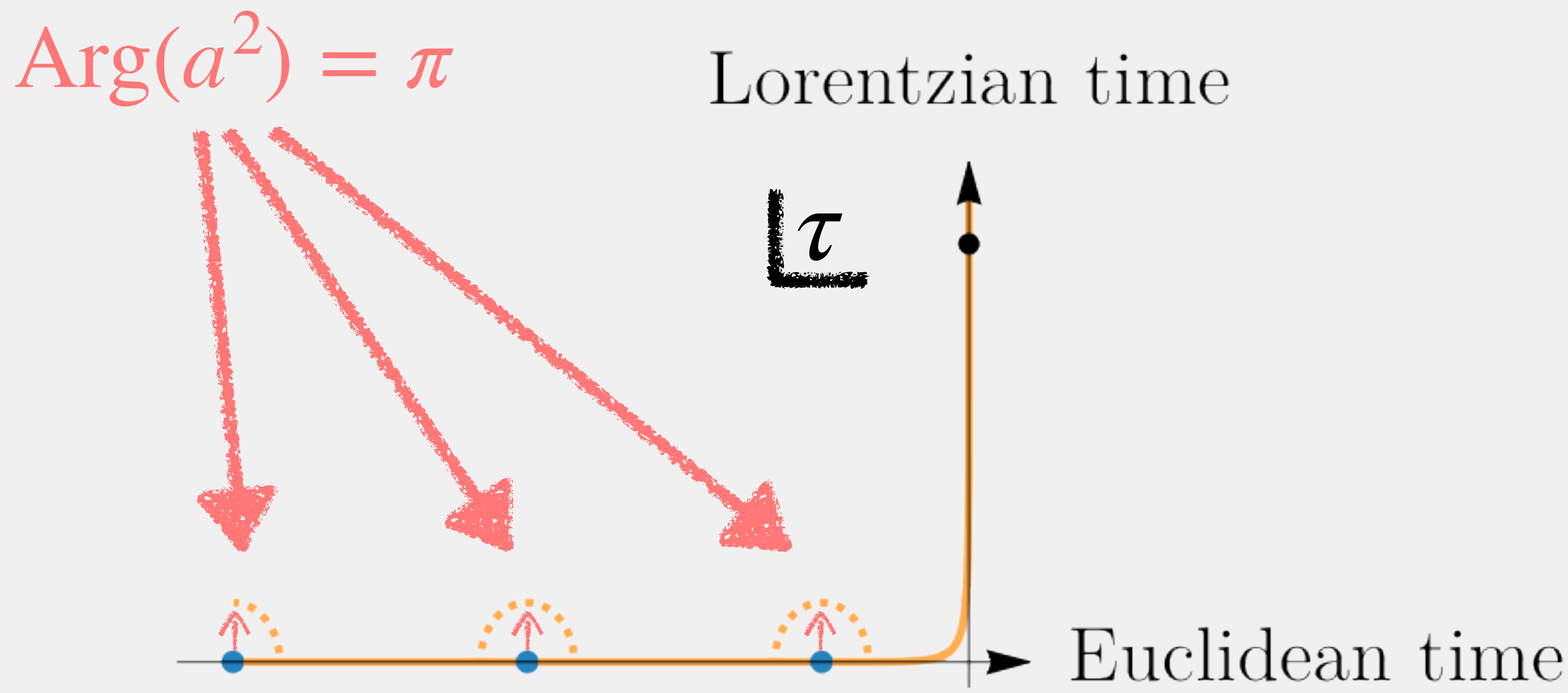

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Jonas-Lehners-JQ [JHEP08(2022)284]

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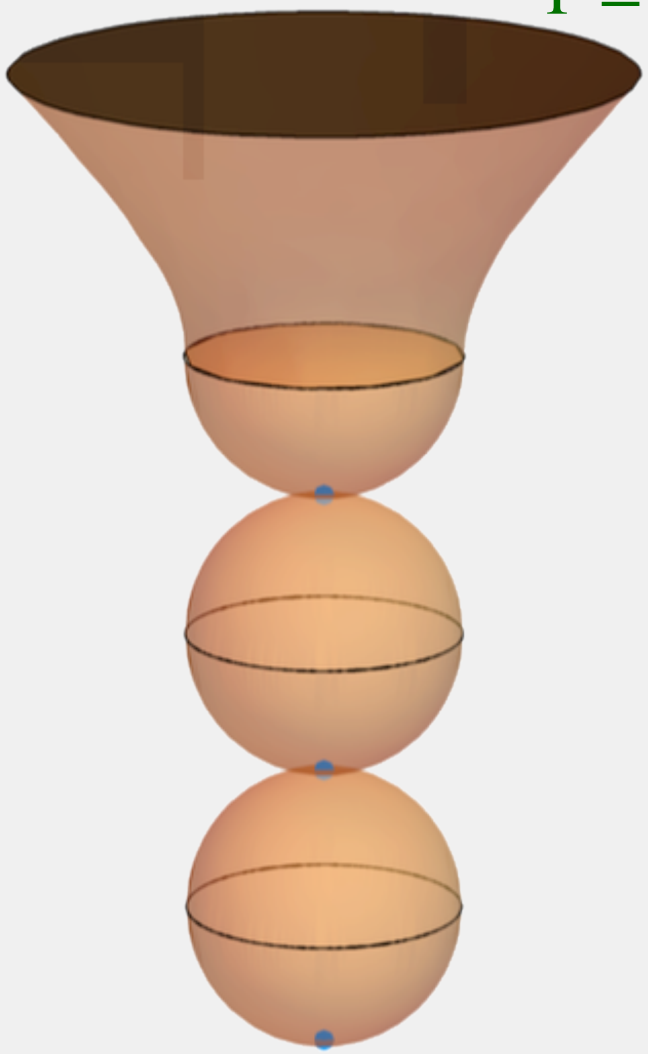
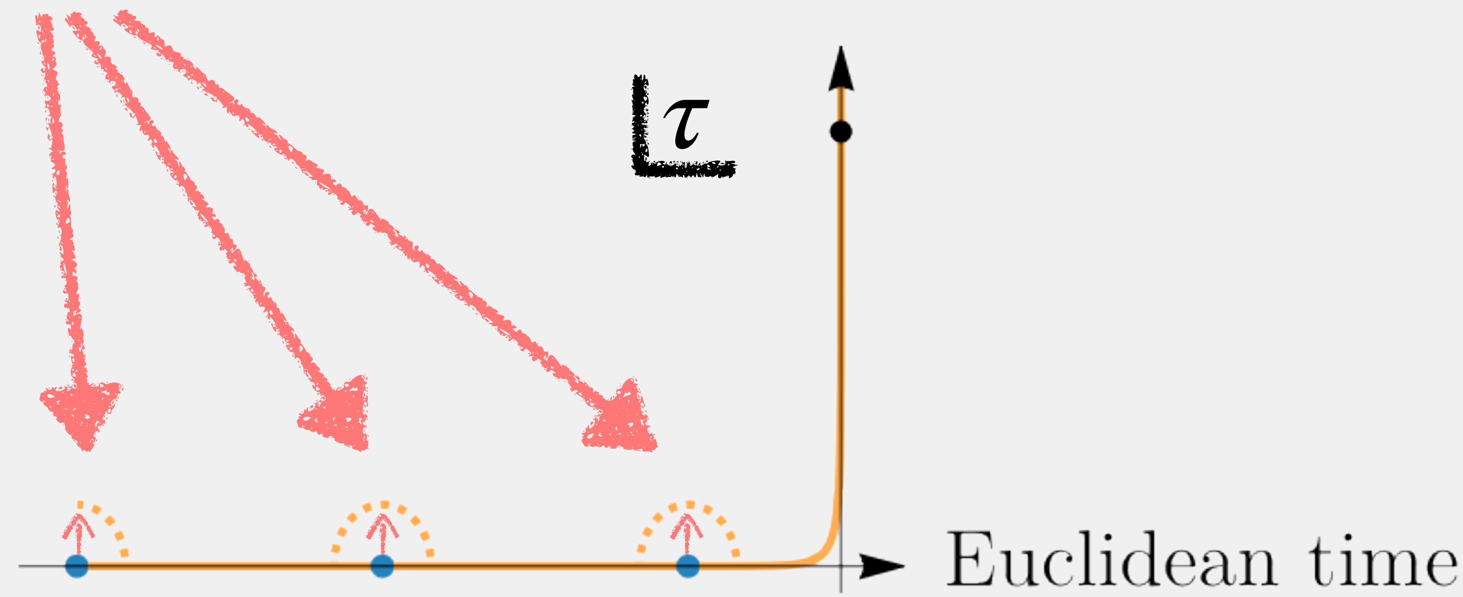


$$\mathrm{Arg}(a^2) = \pi$$



$$\mathrm{d}s^2 = \tau'(u)^2 \mathrm{d}u^2 + a^2(\tau(u))\mathrm{d}\Omega^2_{(3)}\,, \quad a(\tau) = \frac{\sin(H\tau)}{H}$$

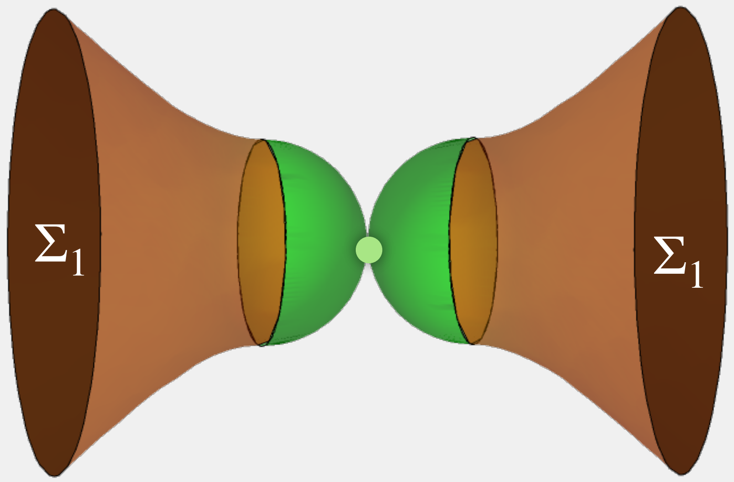
$$\Sigma = |\,\mathrm{Arg}(\tau'^2)\,| + 3\,|\,\mathrm{Arg}(a^2)\,| \geq 3\,|\,\mathrm{Arg}(a^2)\,| \overset{!}{=} 3\pi \nless \pi$$

→ non-allowable

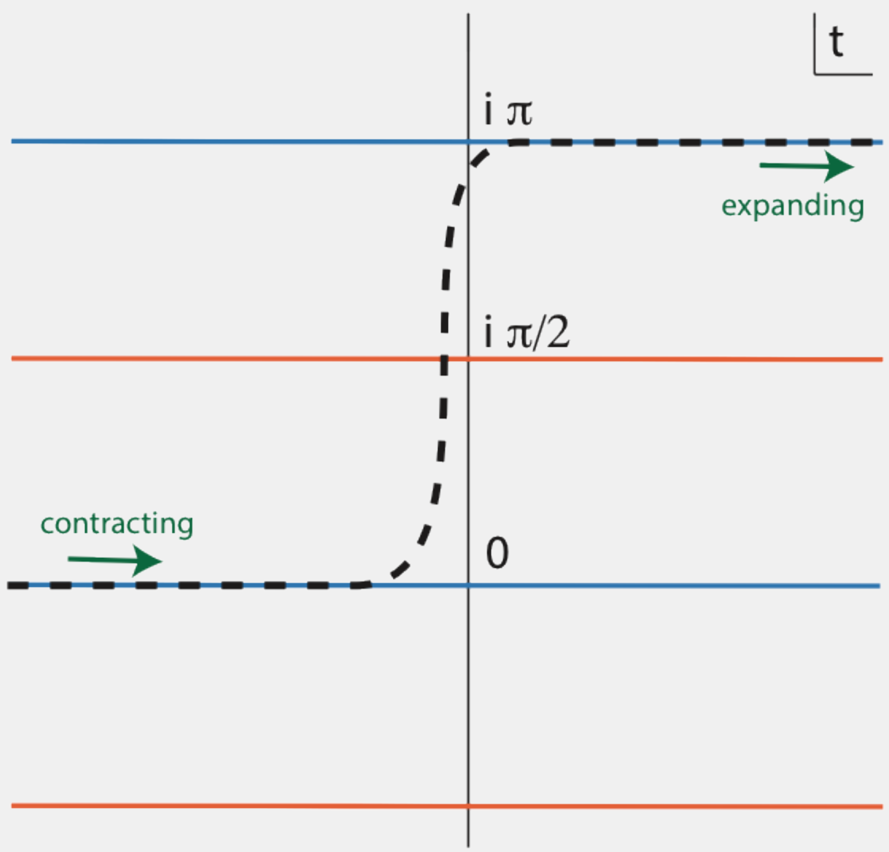
Jonas-Lehners-JQ [JHEP08(2022)284]

Quantum bounces are similarly non-allowable

$$\Psi = \int_{\Sigma_1}^{\Sigma_1} \mathscr{D}g \, e^{\frac{i}{\hbar}S[g]}$$



$$a(t) = \cosh(t)$$

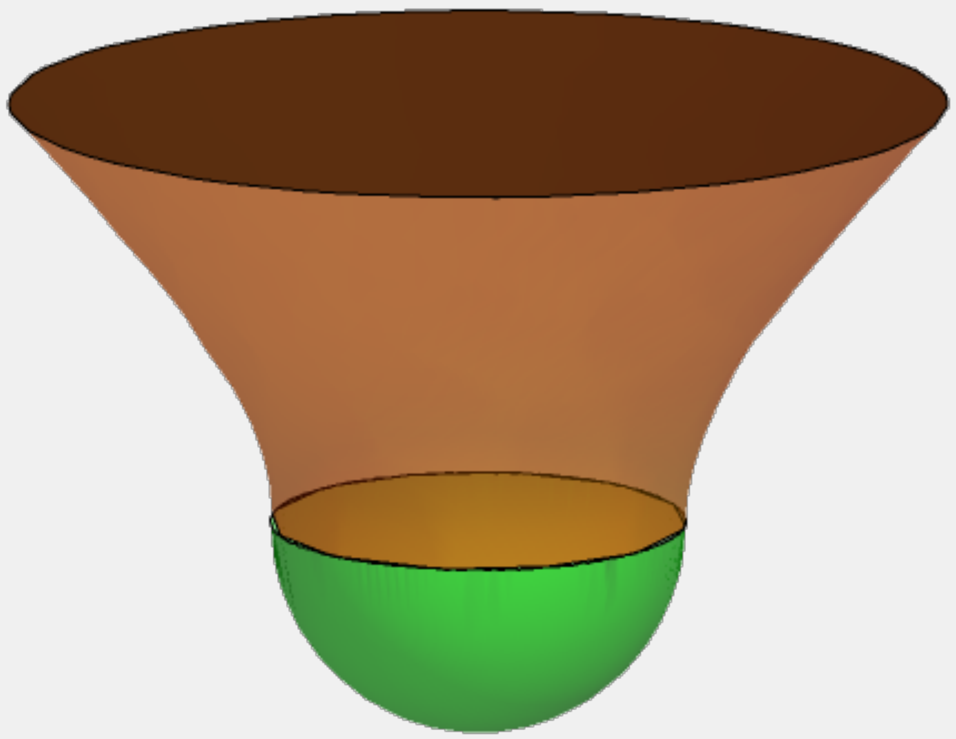


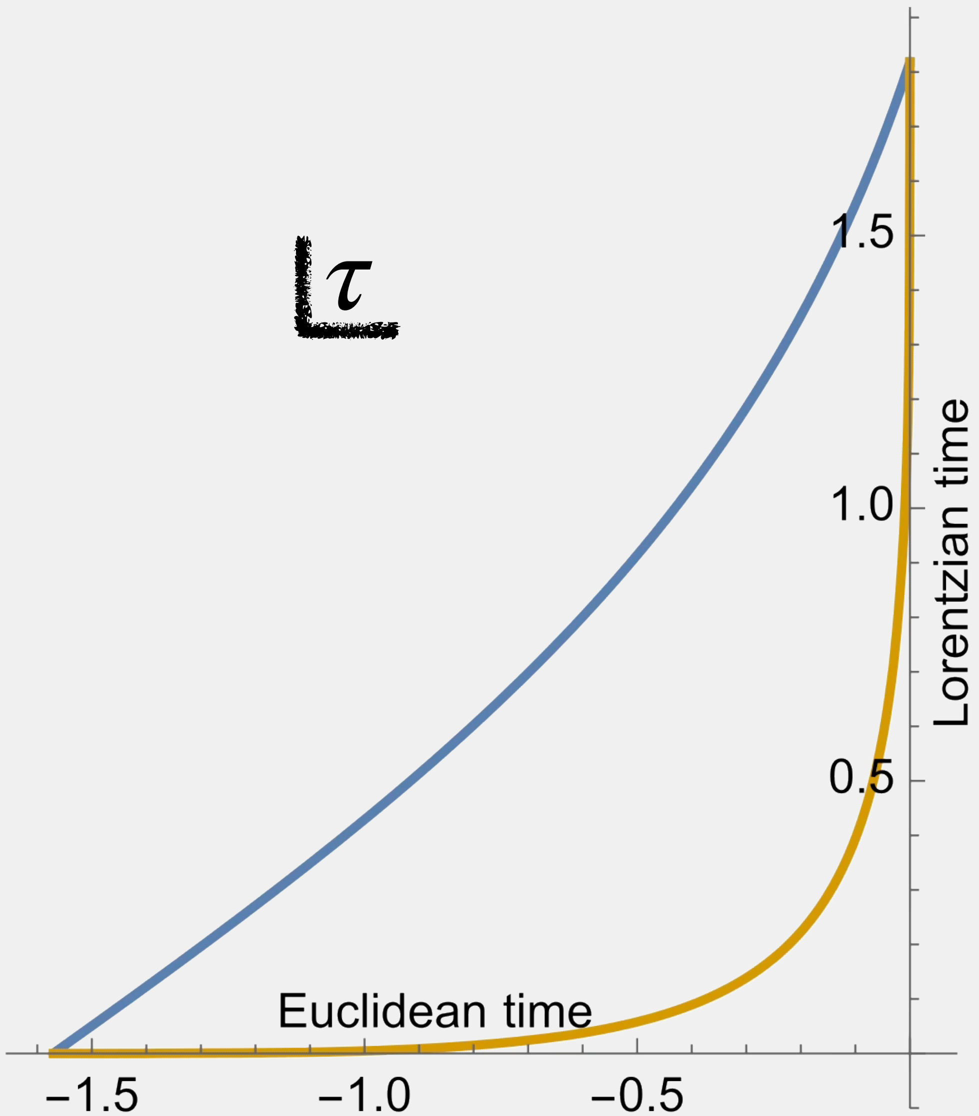
$a(t) \in \mathbb{R}$

$a(t) \in \mathbb{C} \backslash \mathbb{R}$

$a(t) \in \mathbb{R}$

Consider the no-boundary saddle

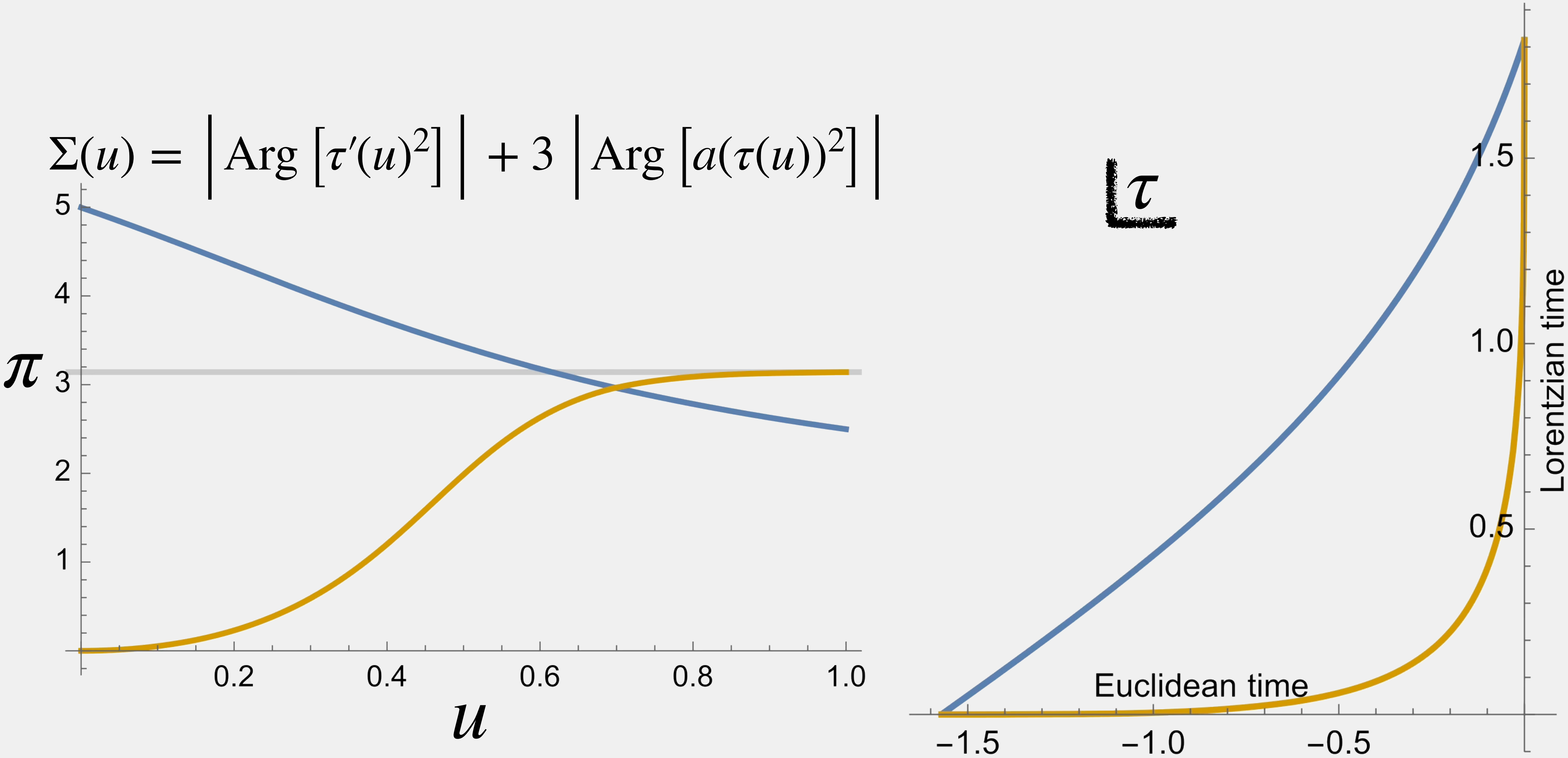




$$\mathrm{d}s^2 = \tau'(u)^2 \mathrm{d}u^2 + a^2(\tau(u))\mathrm{d}\Omega^2_{(3)}, \quad a(\tau) = \frac{\sin(H\tau)}{H}$$

Consider the no-boundary saddle

$$\Sigma(u) = \left| \operatorname{Arg}\left[\tau'(u)^2\right] \right| + 3 \left| \operatorname{Arg}\left[a(\tau(u))^2\right] \right|$$



$$\mathrm{d}s^2 = \tau'(u)^2 \mathrm{d}u^2 + a^2(\tau(u))\mathrm{d}\Omega^2_{(3)}, \quad a(\tau) = \frac{\sin(H\tau)}{H}$$

Let's solve the equations of motion on a complex path where $\Sigma(u) = \pi$

Lehners [2209.14669]
Hertog-Janssen-Karlsson [2305.15440]
Lehners-JQ [PLB 850(2024)138488]

Let's solve the equations of motion on a complex path where $\Sigma(u) = \pi$

$$
\mathrm{d}s^2 = \mathrm{d}\tau^2 + a(\tau)^2 \mathrm{d}\Omega_{(3)}^2 \Rightarrow \begin{cases} a_{,\tau\tau} + \dfrac{a}{3}\left((\phi_{,\tau})^2 + V(\phi)\right) = 0 \\ \phi_{,\tau\tau} + 3\dfrac{a_{,\tau}}{a}\phi_{,\tau} - V_{,\phi} = 0 \end{cases}
$$

Lehners [2209.14669]
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$\Downarrow$

$$\tau \mapsto \gamma(u)$$

Lehners [2209.14669]
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$$\downarrow$$

$$\tau \mapsto \gamma(u)$$

$$\downarrow$$

$$ds^2 = \gamma'(u)^2 du^2 + a^2 d\Omega_{(3)}^2 \Rightarrow \begin{cases} a'' - \frac{\gamma''}{\gamma'}a' + \frac{a}{3}\left(\phi'^2 + \gamma'^2 V(\phi)\right) = 0 \\ \phi'' - \frac{\gamma''}{\gamma'}\phi' + 3\frac{a'}{a}\phi' - \gamma'^2 V_{,\phi} = 0 \end{cases}$$

$$' \equiv \partial_u$$

Lehners [2209.14669]
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$$\downarrow$$

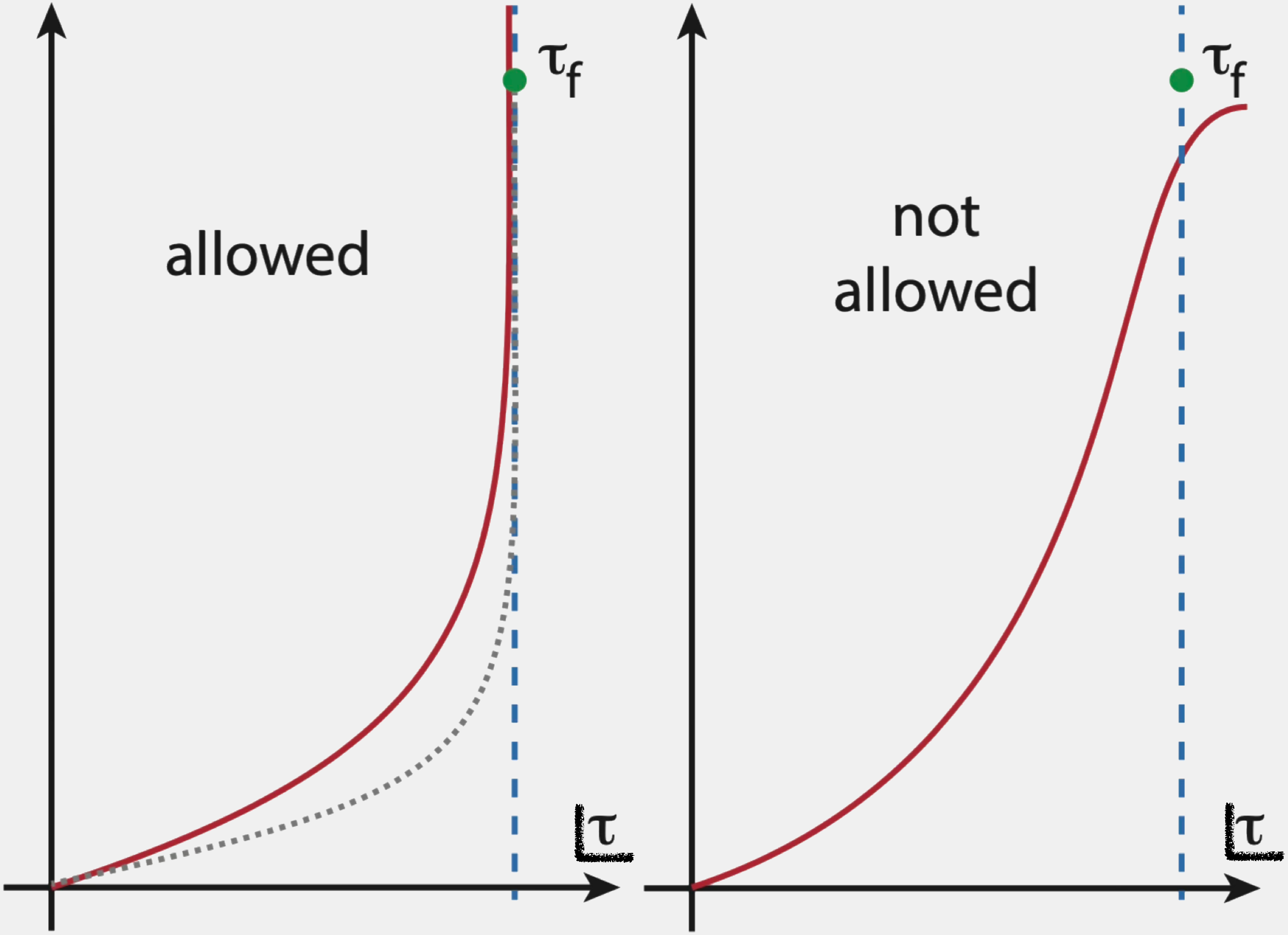
$$\tau \mapsto \gamma(u)$$

$$\downarrow$$

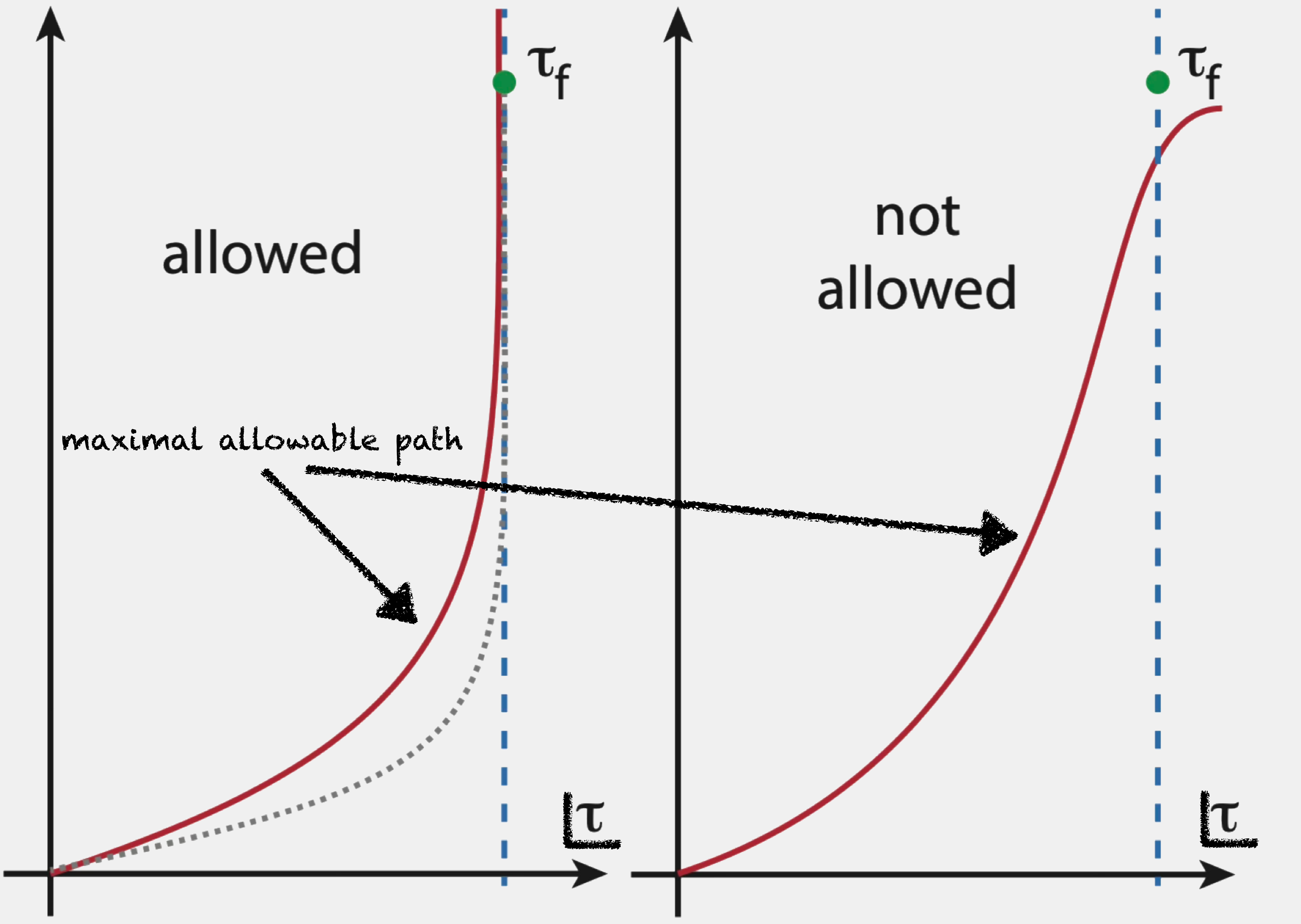
$$\mathrm{d}s^2 = \gamma'(u)^2 \mathrm{d}u^2 + a^2 \mathrm{d}\Omega_{(3)}^2 \Rightarrow \begin{cases} a'' - \frac{\gamma''}{\gamma'}a' + \frac{a}{3}\left(\phi'^2 + \gamma'^2 V(\phi)\right) = 0 \\ \phi'' - \frac{\gamma''}{\gamma'}\phi' + 3\frac{a'}{a}\phi' - \gamma'^2 V_{,\phi} = 0 \end{cases}$$

$$' \equiv \partial_u$$

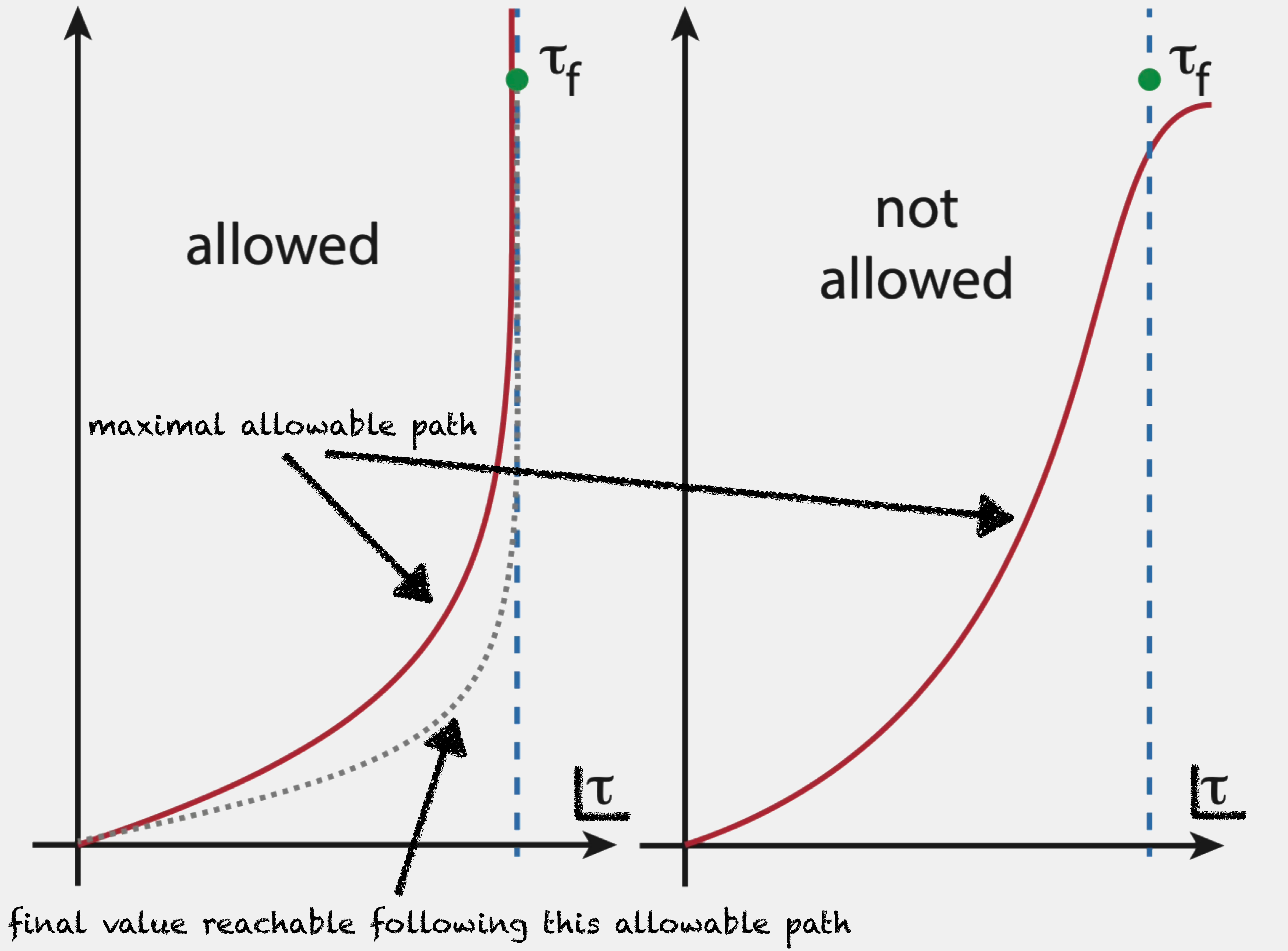
$$+ \qquad \gamma' = i\left(\frac{a^*}{a}\right)^{3/2} \Rightarrow \left|\mathrm{Arg}\left(\gamma'^2\right)\right| + 3\left|\mathrm{Arg}(a^2)\right| = \pi$$



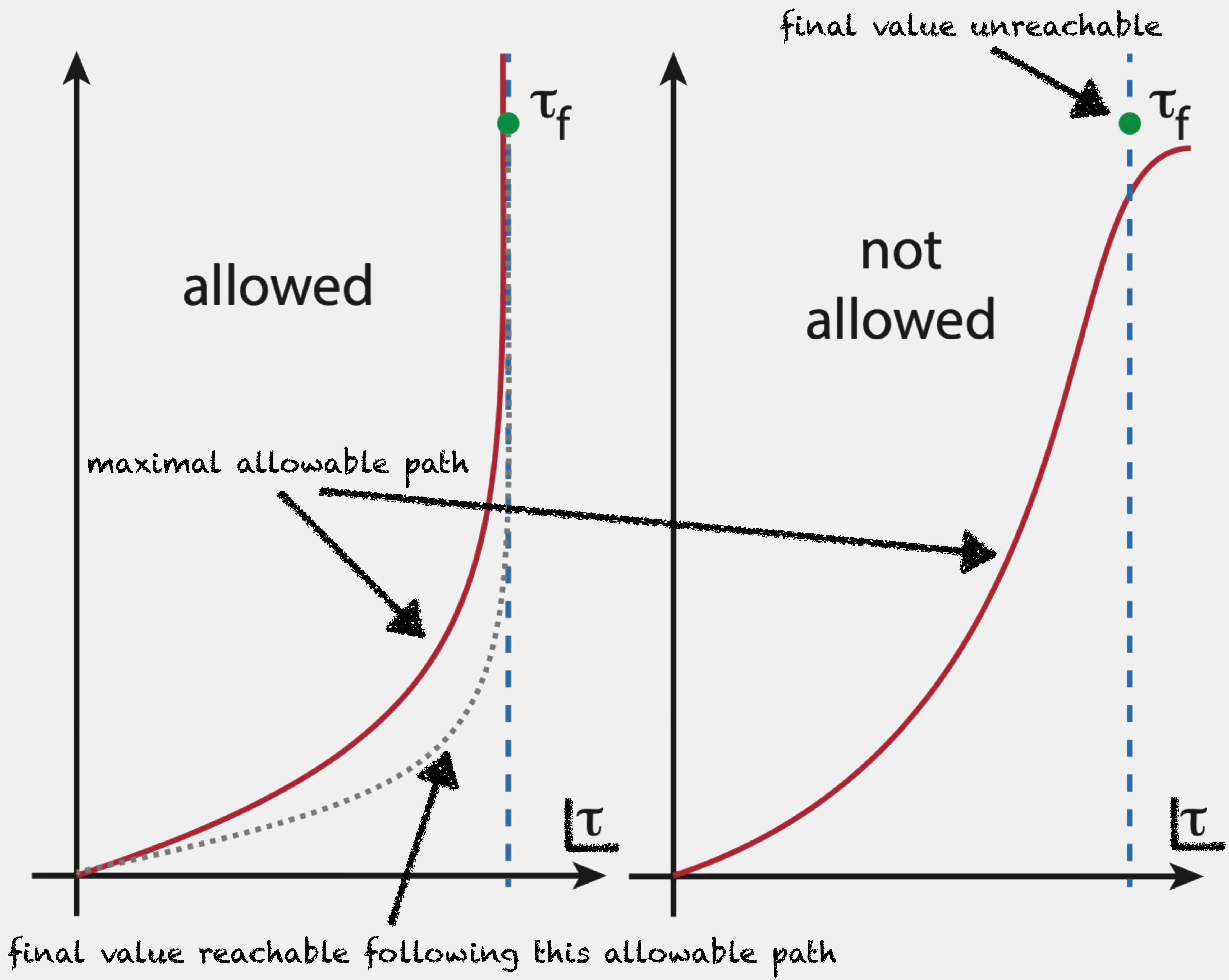
Lehners-JQ [PLB 850(2024)138488]



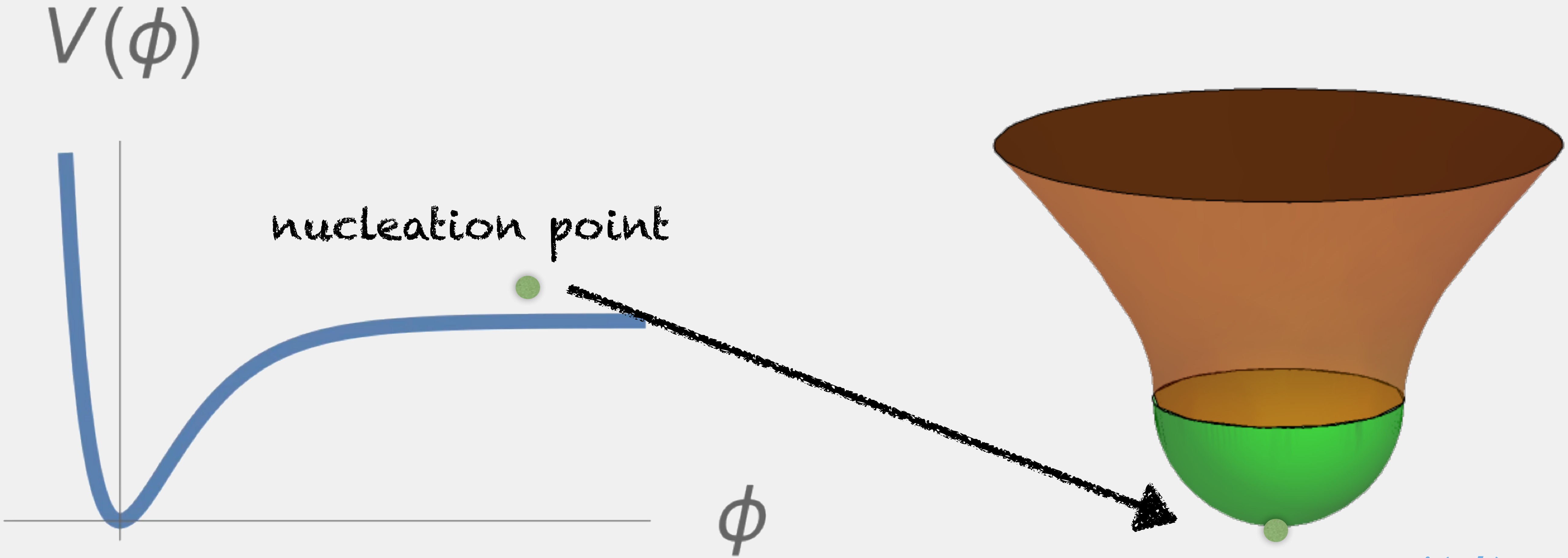
Lehners-JQ [PLB 850(2024)138488]



Lehners-JQ [PLB 850(2024)138488]



Lehners-JQ [PLB 850(2024)138488]

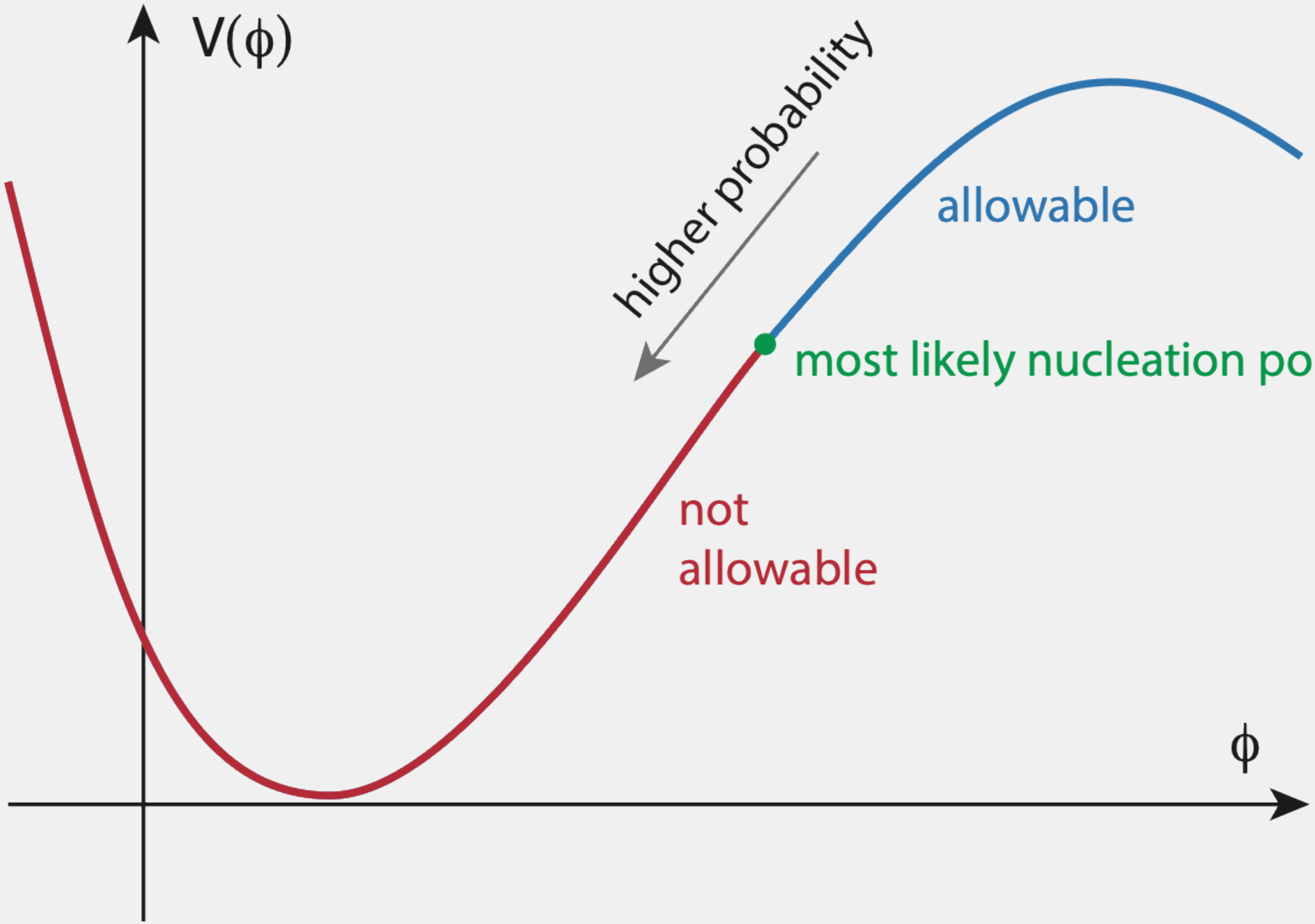


$$\text{Prob}(\phi_{\text{i}}) \sim \exp\left(\frac{24\pi^2}{\hbar V(\phi_{\text{i}})}\right)$$

$$\phi_{\text{SP}} \approx \phi_{\text{i}} - i\frac{\pi}{2}\frac{V'(\phi)}{V(\phi)}$$

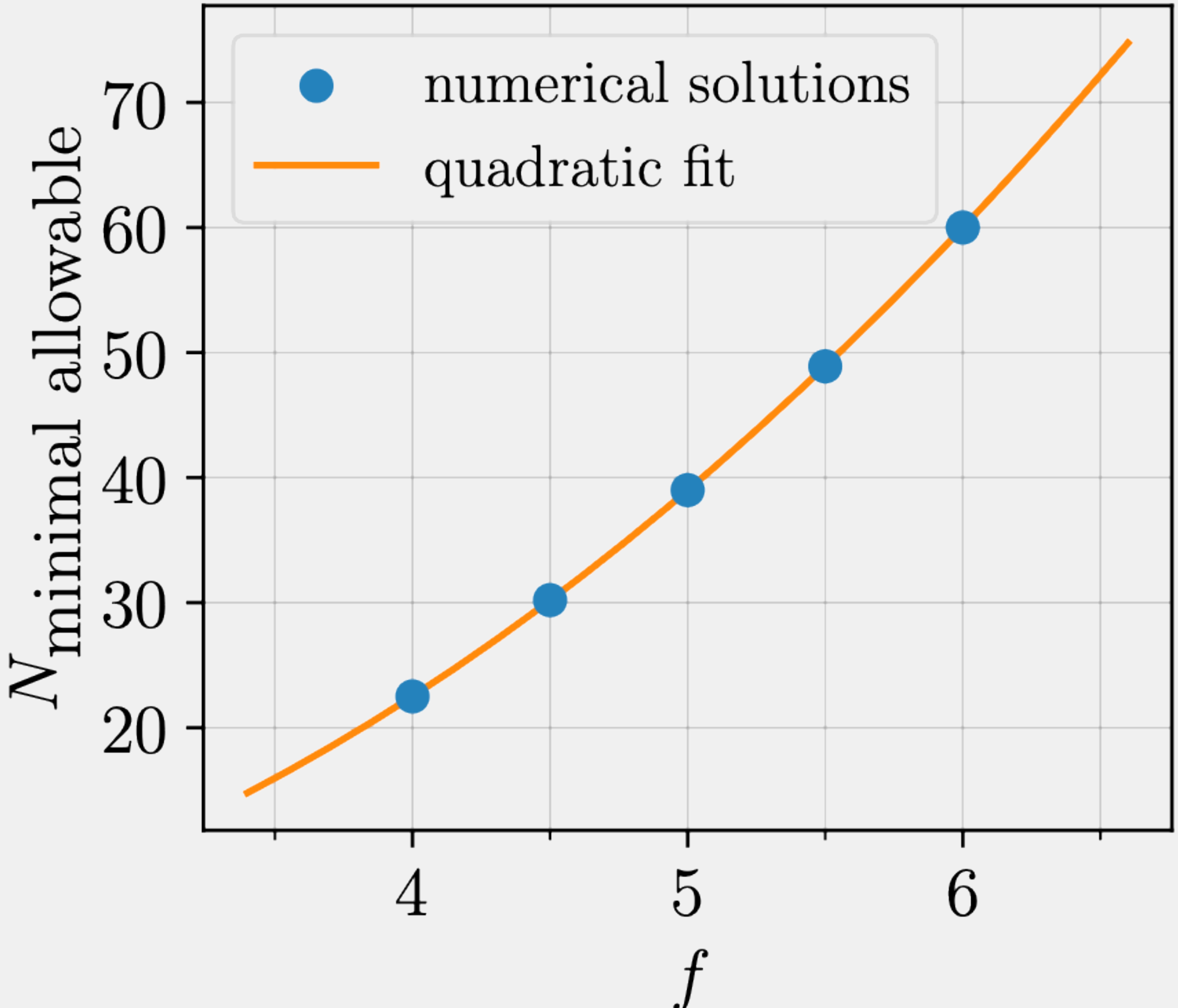
- Nucleation probability favours nucleation low on the potential

- Kontsevich-Segal disfavours ϕ_{SP} too complex,
  so favours nucleation on flatter potentials



Lehners-JQ [PLB 850(2024)138488]

$$V(\phi) = V_0\left(1 + \cos(\phi/f)\right)\,, \qquad N = \ln(a_{\rm end}/a_{\rm nucl})$$



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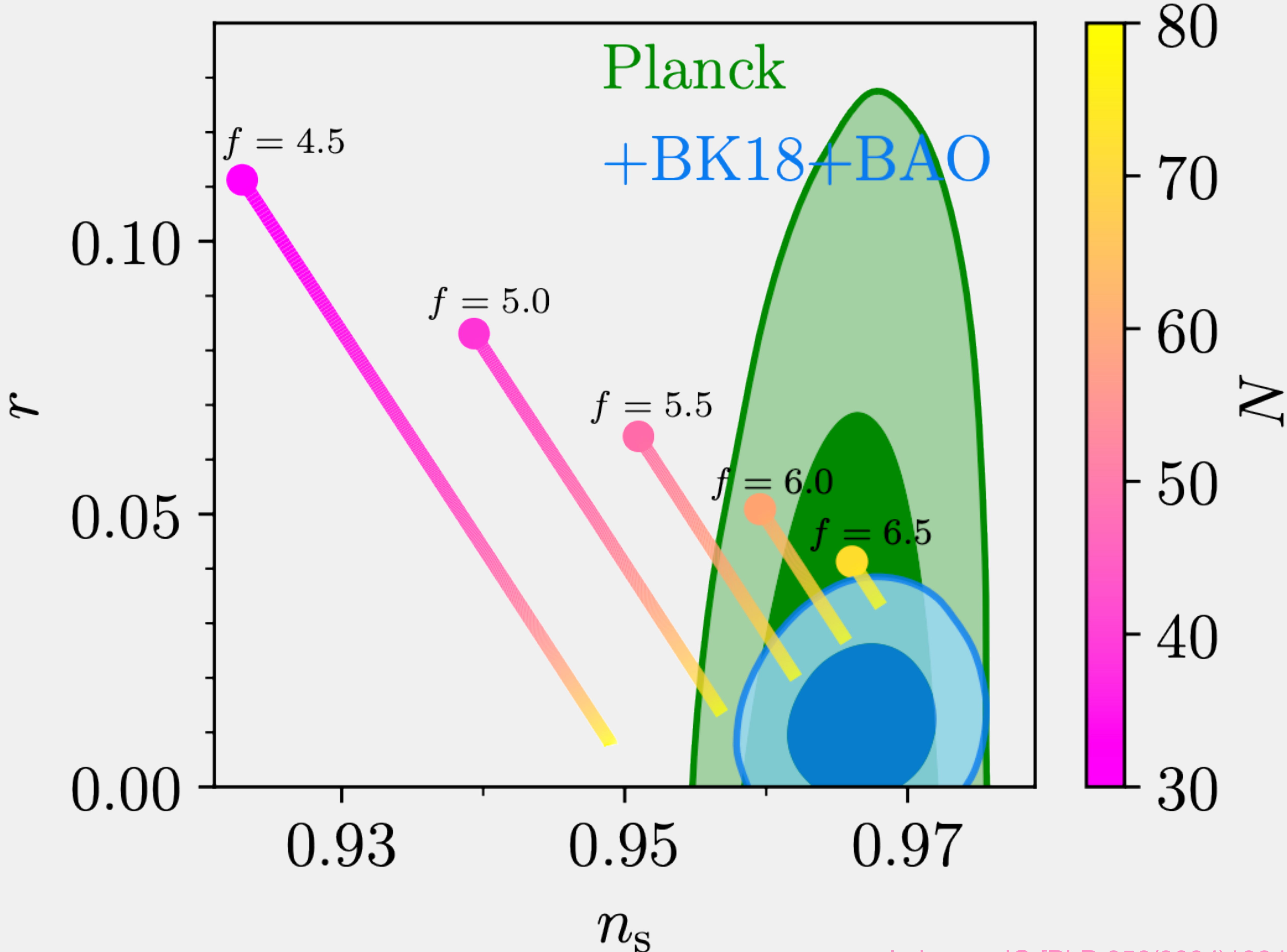


Lehners-JQ [PLB 850(2024)138488]

Hints of a minimal inflationary duration?

Hints of a small universe?

$$a_0 = \frac{1}{H_0\sqrt{-\Omega_{K,0}}} = \frac{10}{H_0}\sqrt{\frac{-0.01}{\Omega_{K,0}}}$$

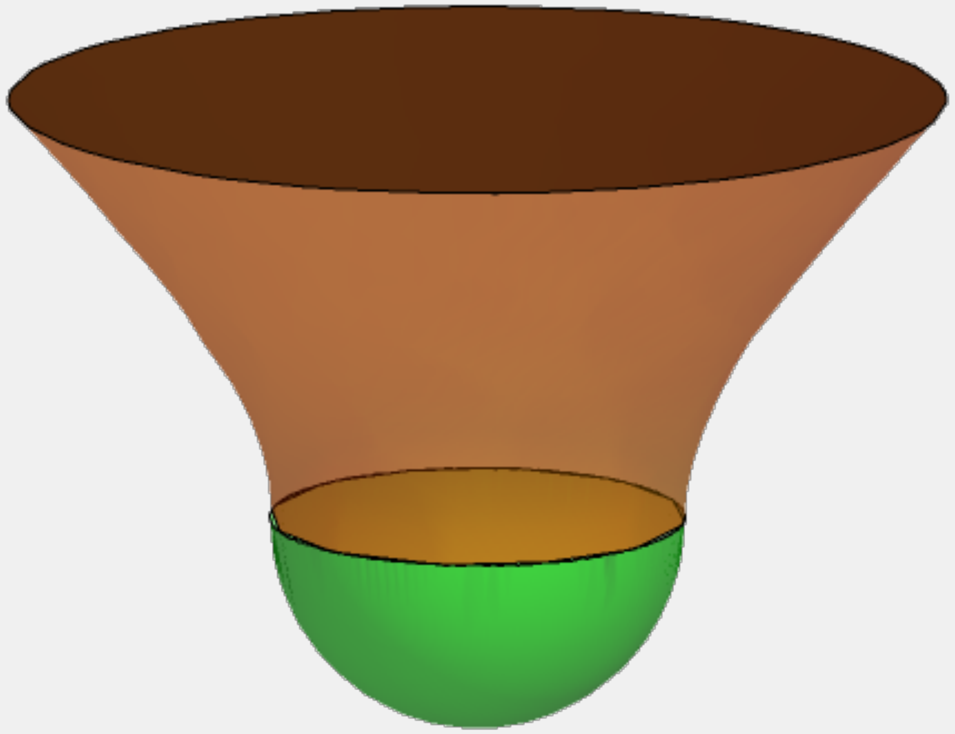
$$N_{\mathrm{infl,min}} \approx 62 + \ln\left(\frac{T_{\mathrm{reh}}}{10^{15}\,\mathrm{GeV}}\right) - \frac{1}{2}\ln\left(\frac{\Omega_{K,0}}{-0.01}\right)$$

$$N_{\mathrm{infl,min}} \approx 34 + \ln\left(\frac{T_{\mathrm{reh}}}{1\,\mathrm{TeV}}\right) - \frac{1}{2}\ln\left(\frac{\Omega_{K,0}}{-0.01}\right)$$

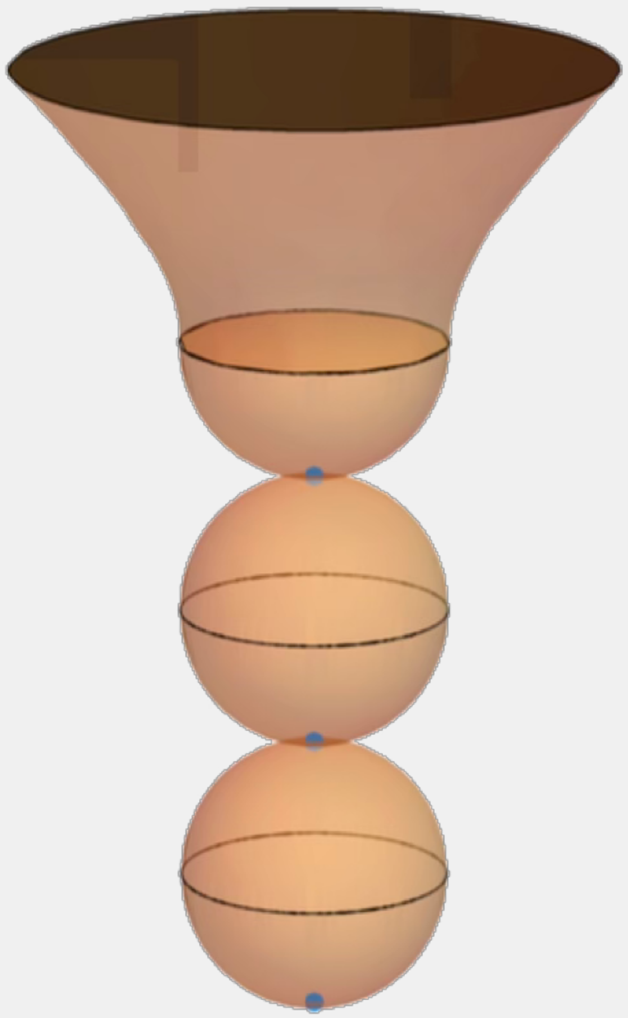
Further takeaways

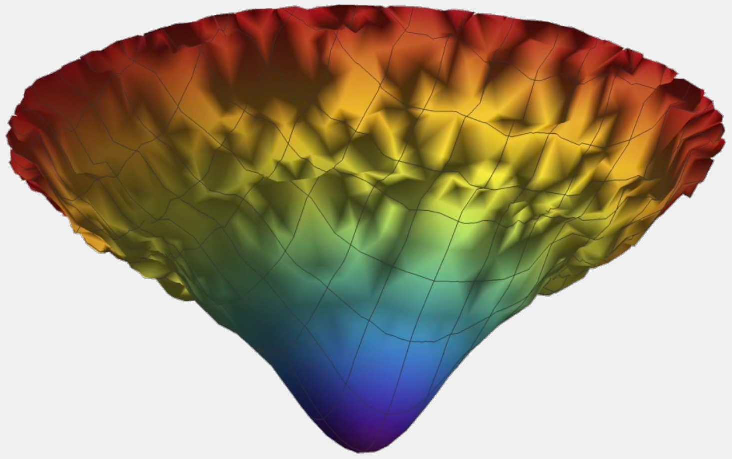
- The Kontsevich-Segal bound appears to be a reasonable criterion on complex metrics
- It is only satisfied for sufficiently long inflation
- With the weighting of the wave function, it suggests minimal inflation

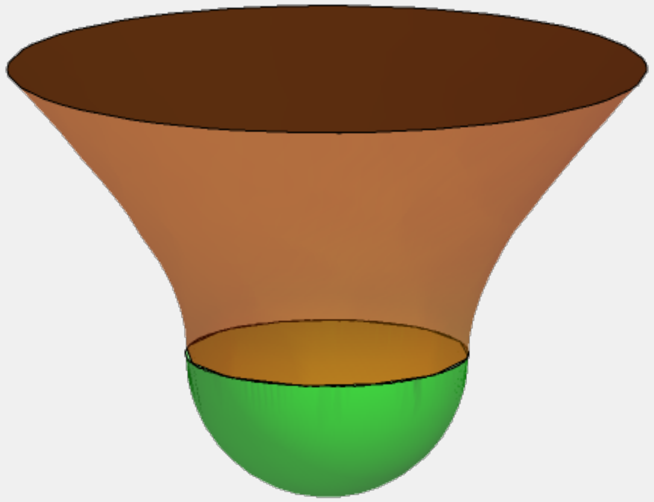
Summary so far

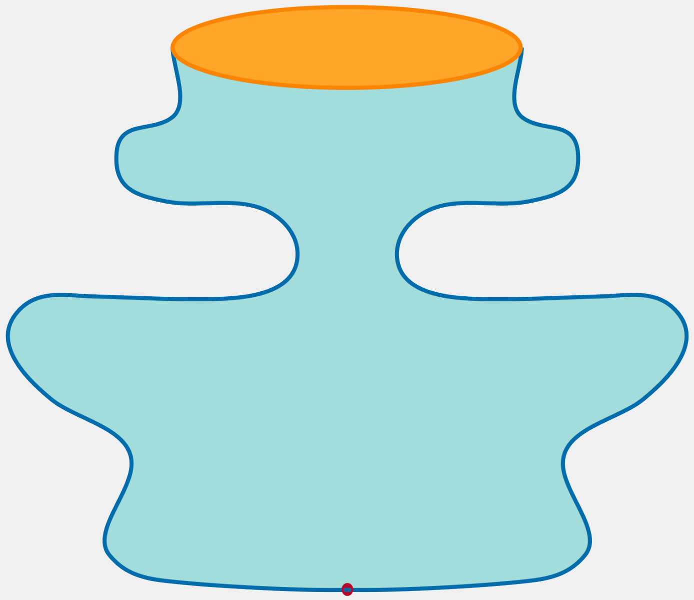


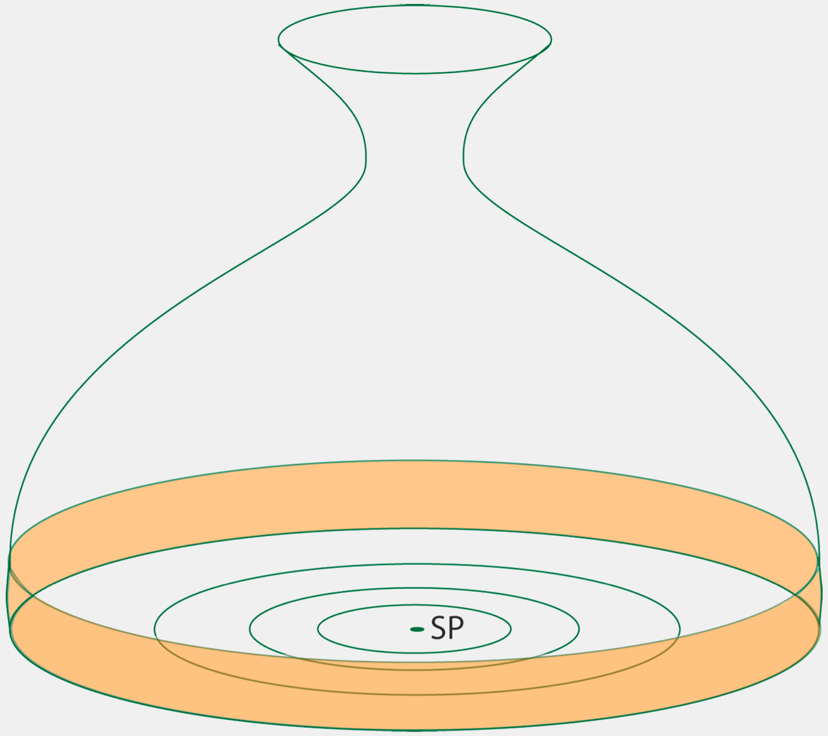
Summary so far





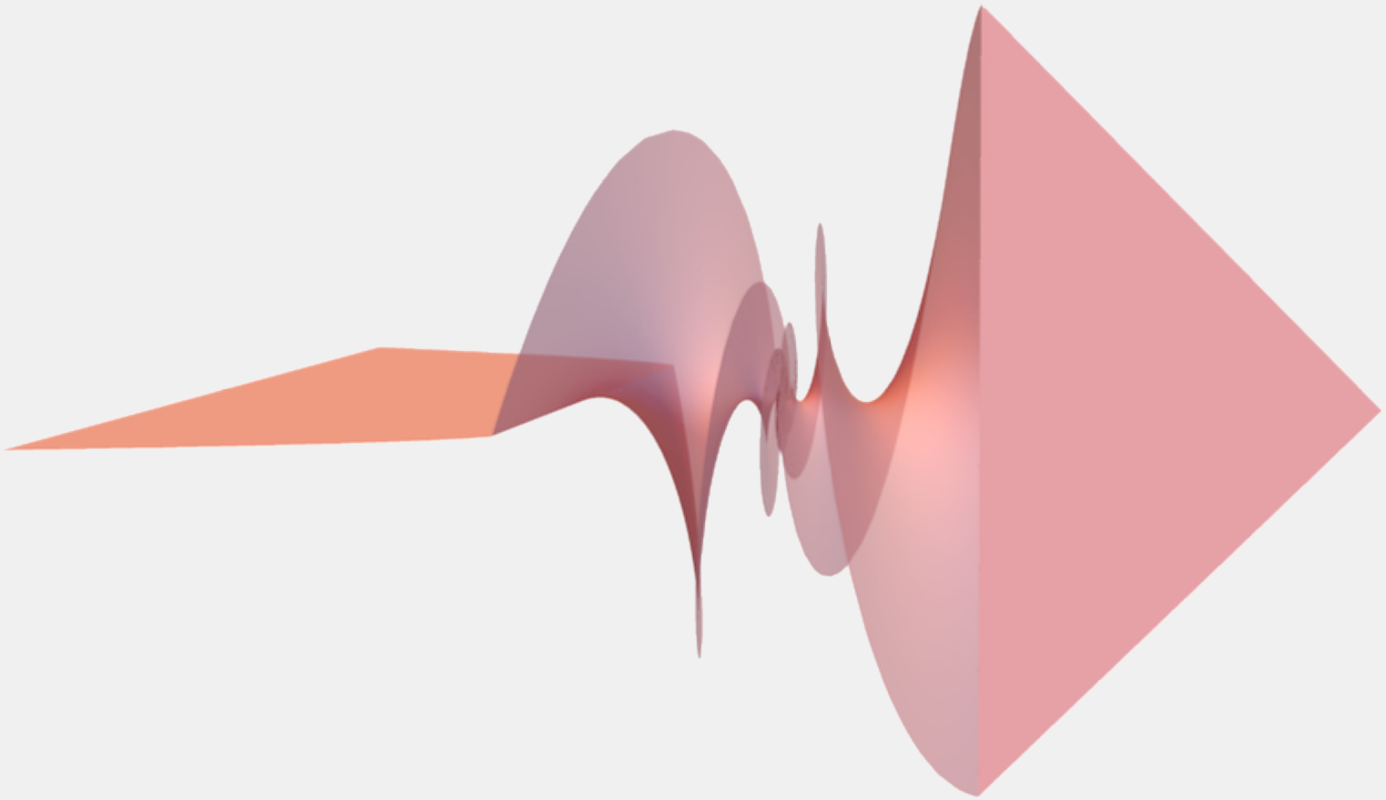


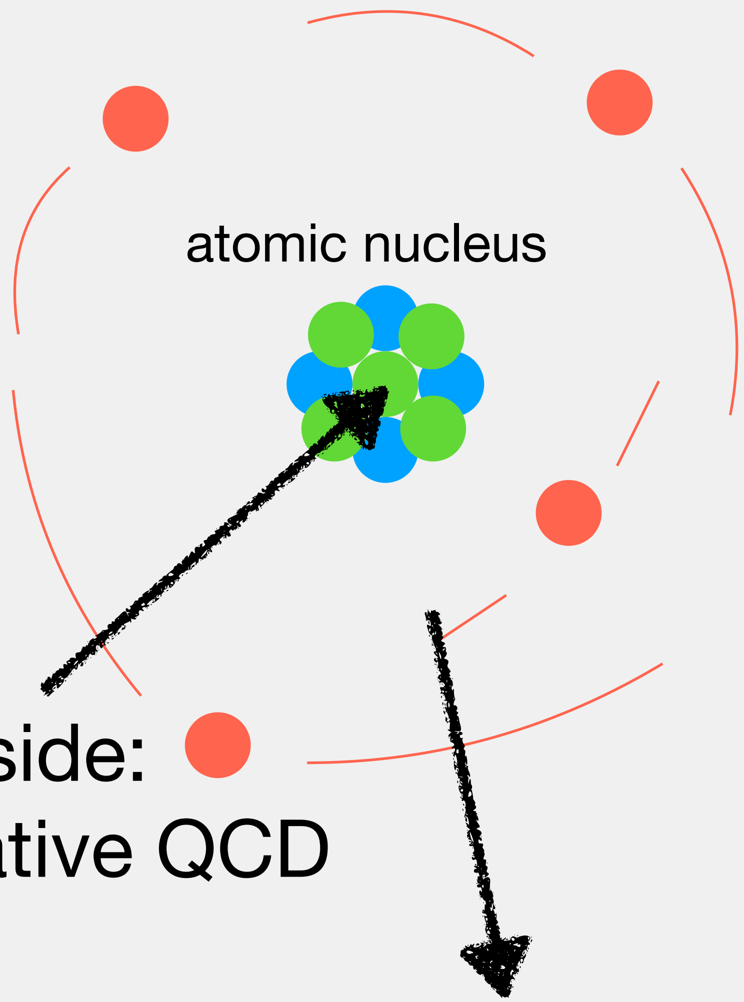




Part III

Complex “black holes”?

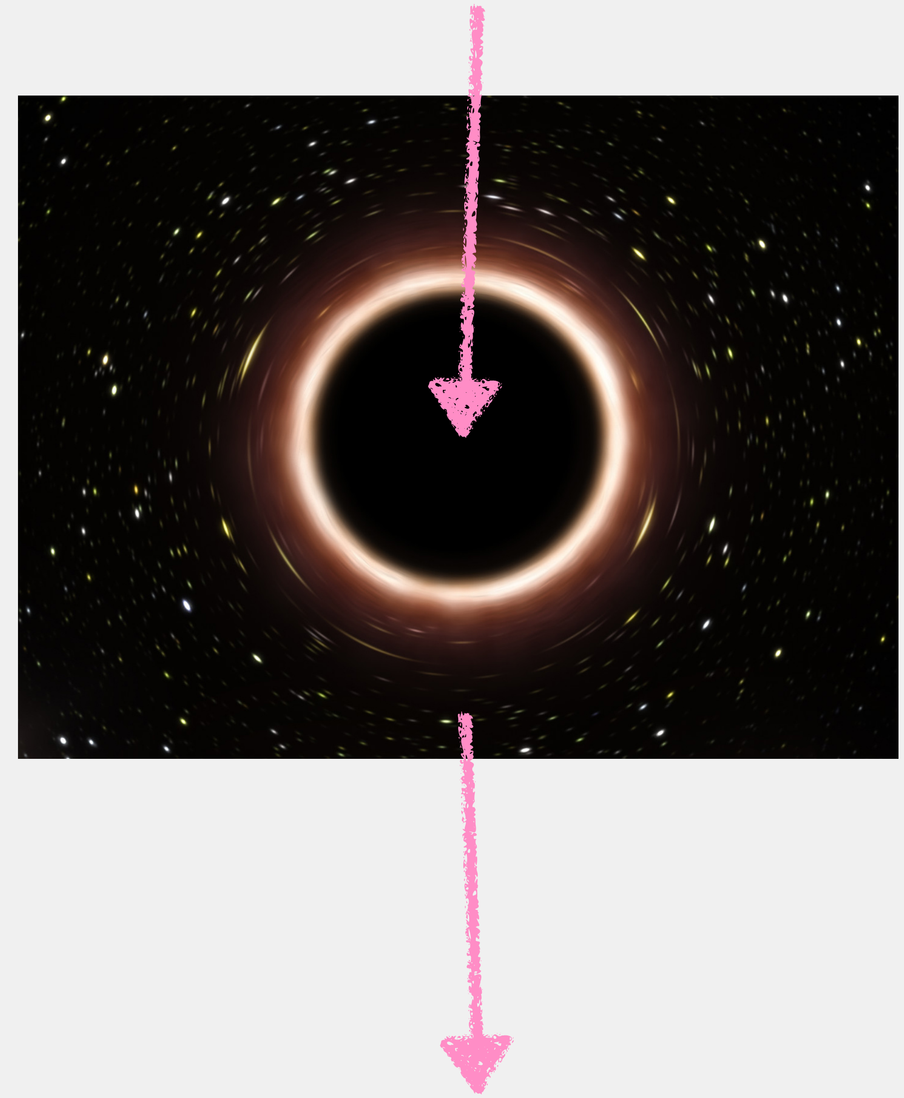




Deep inside:
perturbative QCD

Far away:
perturbative QED

Deep inside: ??



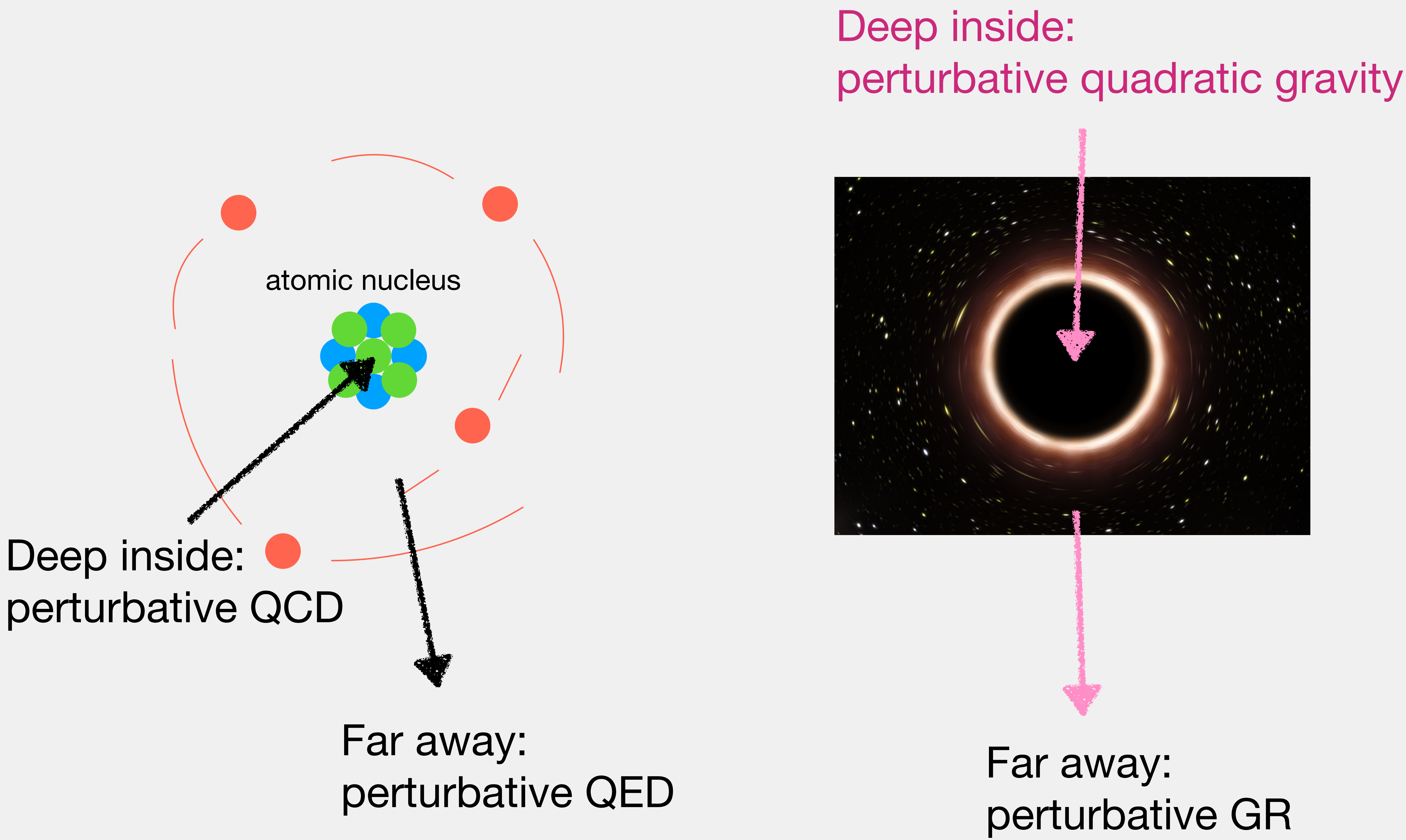
Far away:
perturbative GR

$c = \hbar = 1$

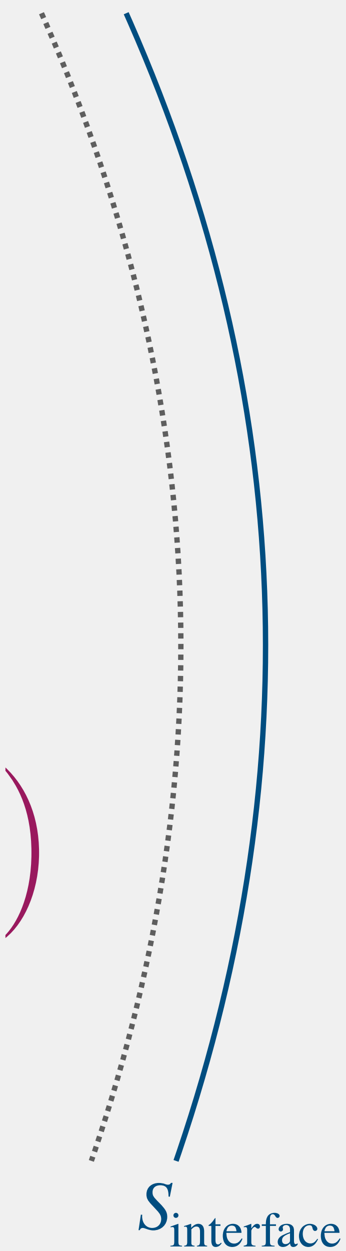
Quadratic gravity

$$S_{\text{quad grav}} = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{1}{16\pi G} R + \frac{\omega}{3\sigma} R^2 - \frac{1}{2\sigma} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} \right)$$

- Renormalizable [Stelle ‘77]
- Asymptotically free UV fixed point [Fradkin-Tseytlin ’82, Avramidi-Barvinsky ’85, Codello, Percacci, Niedermaier, Ohta, Buccio, Donoghue, Menezes, Parente, Zanusso, Kawai ++]
- But ghosts and potential tachyons…


atomic nucleus
Deep inside:
perturbative QCD
Far away:
perturbative QED
Deep inside:
perturbative quadratic gravity
Far away:
perturbative GR

$r = 2GM$ $\quad r = 2GM + \delta$



$r = 0$

Quad Grav near
its UV fixed point

$$S_{\text{pure quad}} = \int d^4x \sqrt{-g} \left(\frac{\omega}{3\sigma} R^2 - \frac{1}{2\sigma} C^2 \right)$$

$r \to \infty$

$$S_{\text{GR}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R$$

$\Rightarrow$ Schwarzschild

$$S_{\text{pure quad}} = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{\omega}{3\sigma} R^2 - \frac{1}{2\sigma} C^2 \right)$$

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -Ar^p \mathrm{d}t^2 + Br^q \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{(2)}$$

$$q = 0\,, \qquad p = \frac{1}{2}\left(1 \mp i\sqrt{15}\right)\,, \qquad B = \frac{3}{8}\left(1 \mp i\sqrt{15}\right)\,, \qquad A \in \mathbb{C}$$

Complex Powerball:

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -Ar^{(1\mp i\sqrt{15})/2}\mathrm{d}t^2 + \frac{3}{8}\left(1 \mp i\sqrt{15}\right)\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{(2)}$$

→ quantum “black hole interior”

Liu-JQ-Afshordi [PRD 111(2025)044031]

Liu-JQ-Afshordi [PRD 111(2025)044031]

$r = 2GM + \delta$

$r = 0$

$r \to \infty$

$$h_{ab}\Big|_r = \begin{pmatrix} -Ar^{(1\mp i\sqrt{15})/2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$h_{ab}\Big|_r = \begin{pmatrix} \frac{2GM}{r} - 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$S_{\text{interface}}$

Matching $\Rightarrow$ $A = \dfrac{\delta}{(2GM + \delta)^{(3\mp i\sqrt{15})/2}}$

Liu-JQ-Afshordi [PRD 111(2025)044031]

$$r = 2GM + \delta$$

$r = 0$

$$h_{ab}\Big|_r = \begin{pmatrix} -\frac{r^{(1\mp i\sqrt{15})/2}\delta}{(2GM+\delta)^{(3\mp i\sqrt{15})/2}} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$h_{ab}\Big|_r = \begin{pmatrix} \frac{2GM}{r} - 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$r \to \infty$

$$S_{\text{quad grav bdry}} = \int \mathrm{d}^3y\,\sqrt{-h}\,\frac{4}{\sigma}n_\mu n_\nu C^{\mu a\nu b}K_{ab}$$

$S_{\text{interface}}$

$$S_{\text{GHY}} = -\frac{1}{8\pi G}\int \mathrm{d}^3y\,\sqrt{-h}\,K$$

Effective “interpolating theory”: $$S_{\text{interface}} = \frac{1}{8\pi\sqrt{G}}\int \mathrm{d}^3y\,\sqrt{-h}\left(\zeta\,{}^{(3)}R + \mathcal{O}(\sqrt{G})\right)$$

Liu-JQ-Afshordi [PRD 111(2025)044031]

$$r = 2GM + \delta$$

$$r = 0$$

$$h_{ab}\Big|_r = \begin{pmatrix} -\frac{r^{(1\mp i\sqrt{15})/2}\delta}{(2GM+\delta)^{(3\mp i\sqrt{15})/2}} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$h_{ab}\Big|_r = \begin{pmatrix} \frac{2GM}{r} - 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$r \to \infty$$

$$S_{\text{quad grav bdry}} = \int d^3y\,\sqrt{-h}\,\frac{4}{\sigma}n_\mu n_\nu C^{\mu a \nu b}K_{ab}$$

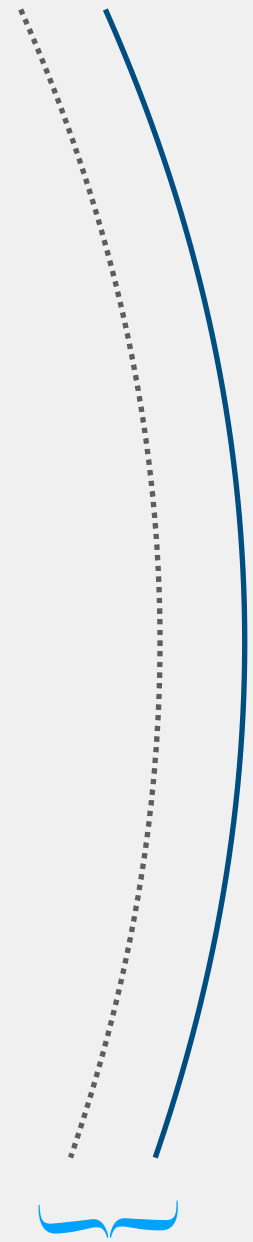
$$S_{\text{GHY}} = -\frac{1}{8\pi G}\int d^3y\,\sqrt{-h}\,K$$

$$S_{\text{interface}} = \frac{\zeta}{8\pi\sqrt{G}}\int d^3y\,\sqrt{-h}\,{}^{(3)}R$$

$$\frac{\delta S_{\text{total boundary}}}{\delta r}\Bigg|_{r=2GM+\delta} = 0 \quad \Rightarrow \quad \delta \simeq \frac{\zeta^2}{8M}$$

$r = 2GM$ $\quad r = 2GM + \delta$

$$\delta \simeq \frac{\zeta^2}{8M}$$



proper distance $\simeq |\zeta| \ell_{\rm Pl}$

Liu-JQ-Afshordi [PRD 111(2025)044031]

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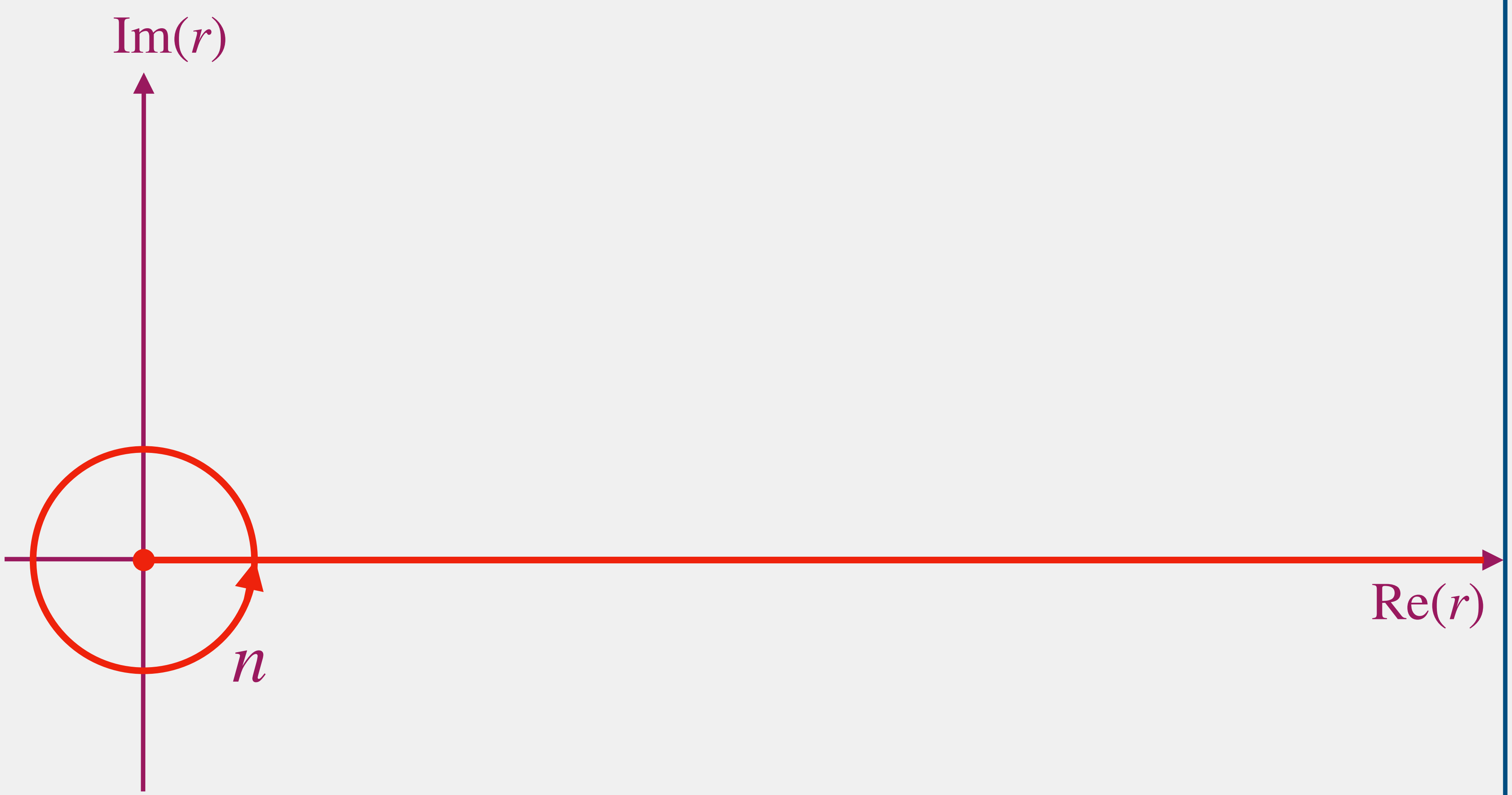

$$r = 2GM + \frac{\zeta^2}{8M}$$

$r \to \infty$

GR: Schwarzschild

$$g_{00} \propto r^{(1\mp i\sqrt{15})/2} = (-1)^n \, |\, r \,|^{1/2} e^{\pm n\pi\sqrt{15} \pm \frac{\sqrt{15}}{2}\mathrm{Arg}(r)} \left(\cos(\omega) + i \sin(\omega)\right)$$

$$\omega = \frac{1}{2}\left(\mathrm{Arg}(r) \mp \sqrt{15}\,\ln|\, r \,|\right)$$

Liu-JQ-Afshordi [PRD 111(2025)044031]

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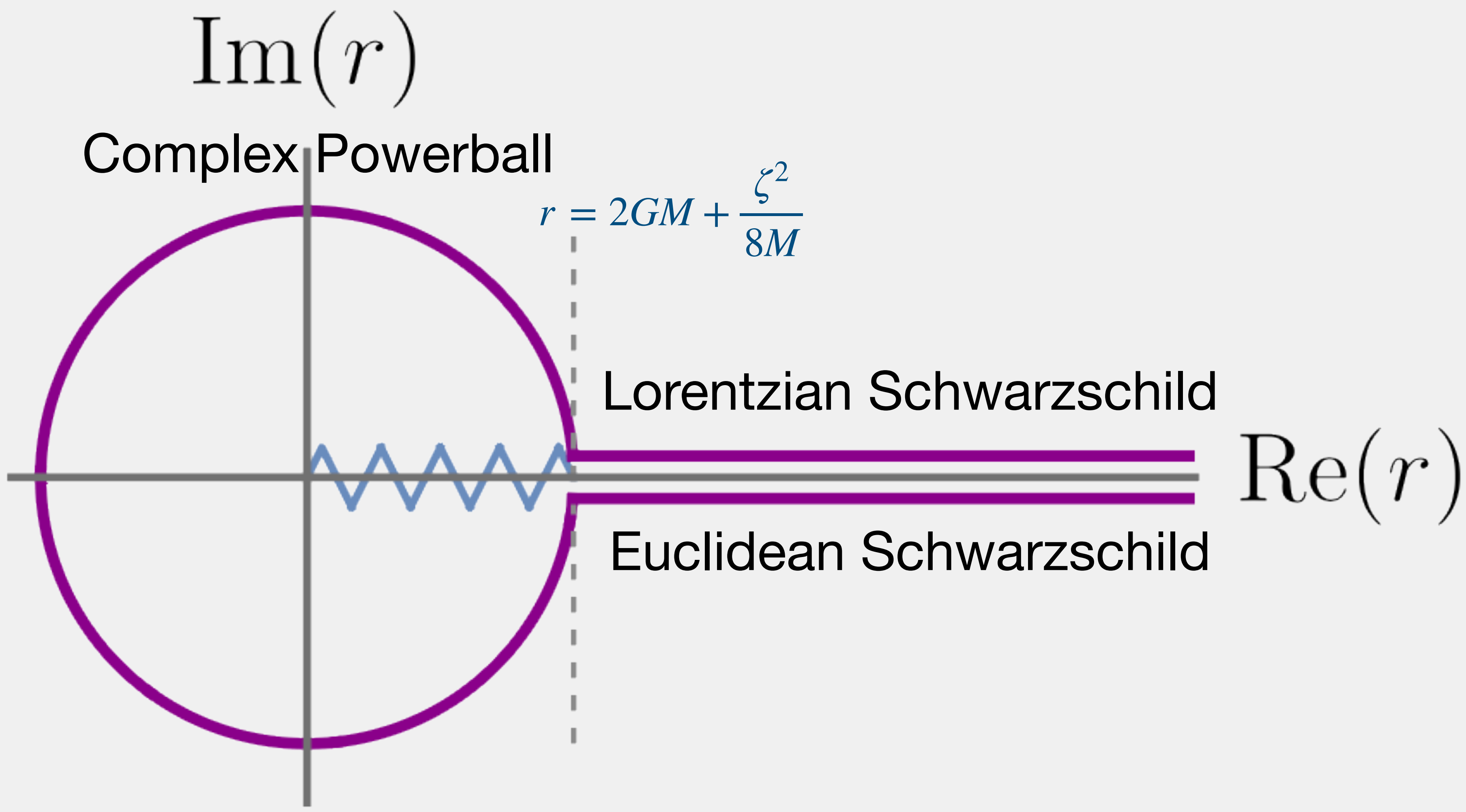

Liu-JQ-Afshordi [PRD 111(2025)044031]

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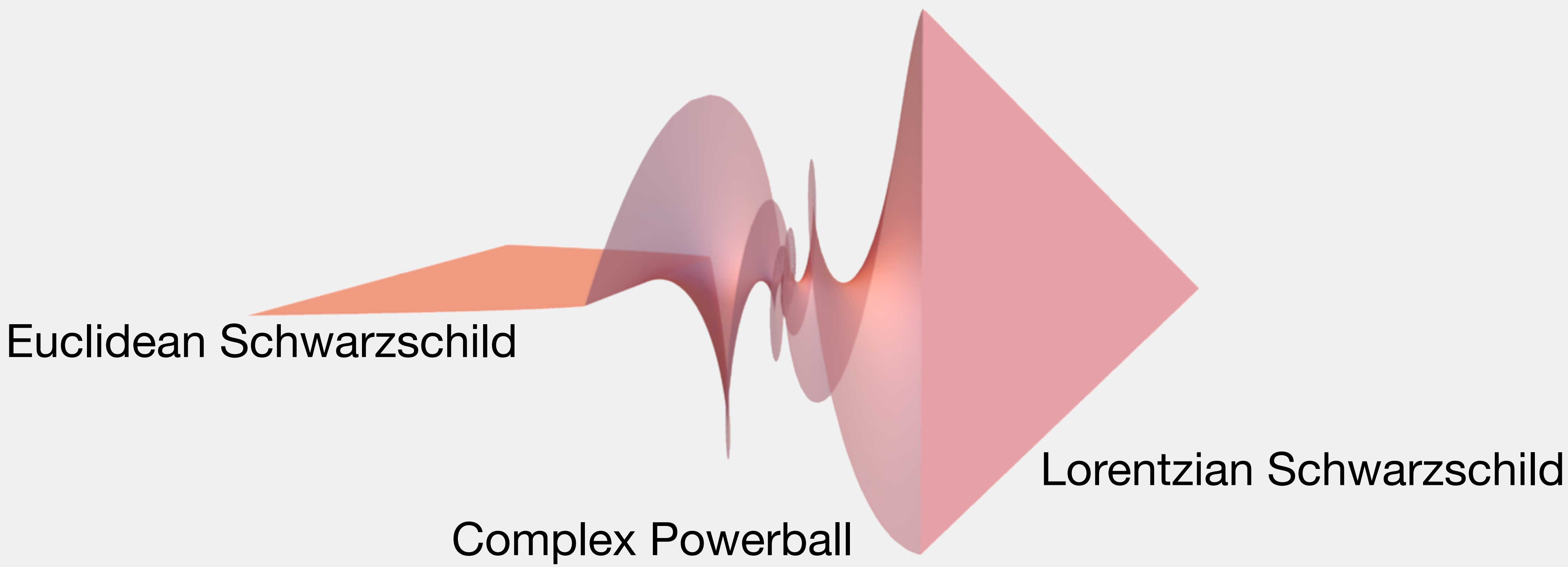
$$\omega = \frac{1}{2}\left(\mathrm{Arg}(r) \mp \sqrt{15}\ln|r|\right)$$



$$\Psi = \int_t^{t+\Delta t} \mathscr{D}g \exp\left(\frac{i}{\hbar}S[g]\right) \sim \exp\left(\frac{i}{\hbar}S_{\text{on-shell}}[g_{\text{powerball}}]\right)$$

$$\Rightarrow \ |\Psi|^2 \sim \exp\left(\eta e^{\pm\eta\pi\sqrt{15}/2}\left(1 - \frac{\zeta^2}{4GM^2}\right)\frac{M\,\Delta t}{\hbar}\right)$$

η → choice of rotation direction

- "Standard Wick rotation" ($\eta = -1$): unstable virtual objects, in accordance with $\Delta E\,\Delta t \geq \hbar/2$
- "Anti-Wick rotation" ($\eta = 1$): exponentially preferred endpoint of gravitational collapse

Liu-JQ-Afshordi [PRD 111(2025)044031]

$$Z = \int_{\tau}^{\tau+\beta} \mathscr{D}g \exp\left(-S_{\rm E}[g]/\hbar\right)$$

$$\Rightarrow \ \ln|Z| \sim -\mathrm{Re}(S_{\rm E}^{\rm on-shell}[g_{\rm powerball}]) \sim -\frac{M\beta}{2}\left(1 - \frac{\zeta^2}{4GM^2} \mp \frac{16\pi e^{\mp\pi\sqrt{15}/2}\zeta}{3\sigma G^{3/2}M^3}\right)$$

⟶ Gibbons-Hawking to leading order, with Planckian corrections

Final takeaways

- Rough toy model of a quantum “horizonless black hole”
- Schwarzschild-like from the exterior, but continuously interpolates toward a pure quadratic gravity complex spacetime (powerball) in the interior
- Has a quantum interpretation, though it depends on the choice of “Wick rotation” (not too different from the no-boundary proposal)

Thank you for your attention!

Questions?

Additional slides

Lightning review:
minisuperspace quantum cosmology

Lehners [Phys.Rept. 2023], Lehners-JQ [JCAP01(2025)027]

$8\pi G_N = c = 1$

$$\Psi = \int \mathscr{D} g_{\mu\nu} \mathscr{D}\phi \exp\left(\frac{i}{2\hbar}\int \mathrm{d}^4x\sqrt{-g}\left(R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi)\right)\right)$$

closed FLRW: $g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + a(t)^2\mathrm{d}\Omega^2_{(3)}$

↓

$$S = 2\pi^2\int \mathrm{d}t\, N\left(-\frac{3}{N^2}a\dot{a}^2 + \frac{1}{2N^2}a^3\dot{\phi}^2 + 3a - a^3V(\phi)\right)$$

↓

$$S = \int \mathrm{d}t\, N\left(\frac{1}{2N^2}G_{AB}\dot{q}^A\dot{q}^B - U(q^A)\right), \qquad q^A = (a,\phi)$$

minisuperspace action:

$$S = \int \mathrm{d}t \, N \left(\frac{1}{2N^2} G_{AB} \dot{q}^A \dot{q}^B - U(q^A) \right)$$

Hamiltonian density:

$$\mathscr{H} = \frac{1}{2} G^{AB} p_A p_B + U = 0$$

Quantization:

$$p_A \longmapsto \hat{p}_A \equiv -i\hbar \nabla_A \,, \qquad \mathscr{H} \longmapsto \hat{\mathscr{H}} = -\frac{\hbar^2}{2} G^{AB} \nabla_A \nabla_B + U$$

Wheeler-DeWitt (WdW):

$$\hat{\mathscr{H}} \Psi = 0$$

weighting + phase:

$$\Psi = \exp\left(\frac{1}{\hbar}\left(\mathscr{W} + i\mathscr{S}\right)\right), \qquad \mathscr{W}, \mathscr{S} \in \mathbb{R}$$

WdW $\Longrightarrow$

$$\begin{cases} \mathcal{O}(\hbar^1): & G^{AB}\nabla_A\nabla_B\mathscr{W} = 0, \quad G^{AB}\nabla_A\nabla_B\mathscr{S} = 0 \\ \mathcal{O}(\hbar^0): & \frac{1}{2}G^{AB}\left(\nabla_A\mathscr{S}\nabla_B\mathscr{S} - \nabla_A\mathscr{W}\nabla_B\mathscr{W}\right) + U = 0, \quad G^{AB}\nabla_A\mathscr{W}\nabla_B\mathscr{S} = 0 \end{cases}$$

WKB: $(\nabla\mathscr{W})^2 \ll (\nabla\mathscr{S})^2 \quad \Longrightarrow \quad \frac{1}{2}(\nabla\mathscr{S})^2 + U \approx 0$

classical Hamilton-Jacobi

weighting + phase:

$$\Psi = \exp\left(\frac{1}{\hbar}\left(\mathscr{W} + i\mathscr{S}\right)\right), \qquad \mathscr{W}, \mathscr{S} \in \mathbb{R}$$

WKB:

$$(\nabla \mathscr{W})^2 \ll (\nabla \mathscr{S})^2 \quad \Longrightarrow \quad \frac{1}{2}(\nabla \mathscr{S})^2 + U \approx 0$$

classical Hamilton-Jacobi

weighting + phase: $\Psi = \exp\left(\frac{1}{\hbar}\left(\mathscr{W} + i\mathscr{S}\right)\right)$, $\mathscr{W}, \mathscr{S} \in \mathbb{R}$

WKB: $(\nabla\mathscr{W})^2 \ll (\nabla\mathscr{S})^2 \implies \frac{1}{2}(\nabla\mathscr{S})^2 + U \approx 0$

classical Hamilton-Jacobi

weighting + phase:

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WKB:

$$(\nabla \mathscr{W})^2 \ll (\nabla \mathscr{S})^2 \quad \Longrightarrow \quad \frac{1}{2}(\nabla \mathscr{S})^2 + U \approx 0$$

classical Hamilton-Jacobi

$$\hat{p}_A \Psi = -i\hbar \nabla_A \Psi \approx (\nabla_A \mathscr{S})\Psi \quad \Longrightarrow \quad p_A \approx \nabla_A \mathscr{S}$$

$\mathscr{S}$ is the “classical” action

$$\Psi = \int \mathscr{D}a \mathscr{D}\phi \, \exp\left(\frac{i}{\hbar} S\right) \approx \exp\left(\frac{i}{\hbar}\left(\mathrm{Re}\, S_{\text{on-shell}} + i\, \mathrm{Im}\, S_{\text{on-shell}}\right)\right)$$

$$\Longrightarrow \quad \mathscr{S} \approx \mathrm{Re}\, S_{\text{on-shell}}, \quad \mathscr{W} \approx -\mathrm{Im}\, S_{\text{on-shell}}$$

weighting + phase:

$$\Psi = \exp\left(\frac{1}{\hbar}\left(\mathscr{W} + i\mathscr{S}\right)\right), \qquad \mathscr{W}, \mathscr{S} \in \mathbb{R}$$

conserved current density:

$$J^A = -\frac{i\hbar}{2}\left(\Psi^* \nabla^A \Psi - \Psi \nabla^A \Psi^*\right) \overset{\text{WdW}}{\Longrightarrow} \nabla_A J^A = 0$$

$$J_A = |\Psi|^2 \nabla_A \mathscr{S} \equiv \rho \nabla_A \mathscr{S}, \qquad \rho = |\Psi|^2 = \exp\left(\frac{2\mathscr{W}}{\hbar}\right)$$

probability density

Jonas-Lehners-JQ [JHEP08(2022)284]

de Sitter classical transitions

Jonas-Lehners-JQ [JHEP08(2022)284]

de Sitter classical transitions

$$g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -\frac{N^2}{q(t)}\mathrm{d}t^2 + q(t)\mathrm{d}\Omega^2_{(3)}$$

Jonas-Lehners-JQ [JHEP08(2022)284]

de Sitter classical transitions

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$$\frac{N\,\mathrm{d}t}{\sqrt{q(t)}} = \mathrm{d}t_{\mathrm{FLRW}}$$

$$q(t) = a(t)^2$$

de Sitter classical transitions

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\frac{N^2}{q(t)}\mathrm{d}t^2 + q(t)\mathrm{d}\Omega^2_{(3)}$$

$$\Psi = \int \mathscr{D}N \int_{q_0}^{q_1} \mathscr{D}q \, e^{\frac{i}{2\hbar}\int \mathrm{d}^4x \sqrt{-g}(R-2\Lambda)}$$

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$$\Psi = \int \frac{\mathrm{d}N}{\sqrt{N}}\, e^{\frac{i}{\hbar}\left(N^3\frac{\Lambda^2}{36} + N\left(3 - \frac{\Lambda}{2}(q_0+q_1)\right) - \frac{3}{4N}(q_1-q_0)^2\right)}$$

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saddles : $N = \frac{3}{\Lambda}\left(\pm\sqrt{\frac{\Lambda}{3}q_1 - 1} \pm \sqrt{\frac{\Lambda}{3}q_0 - 1}\right)$

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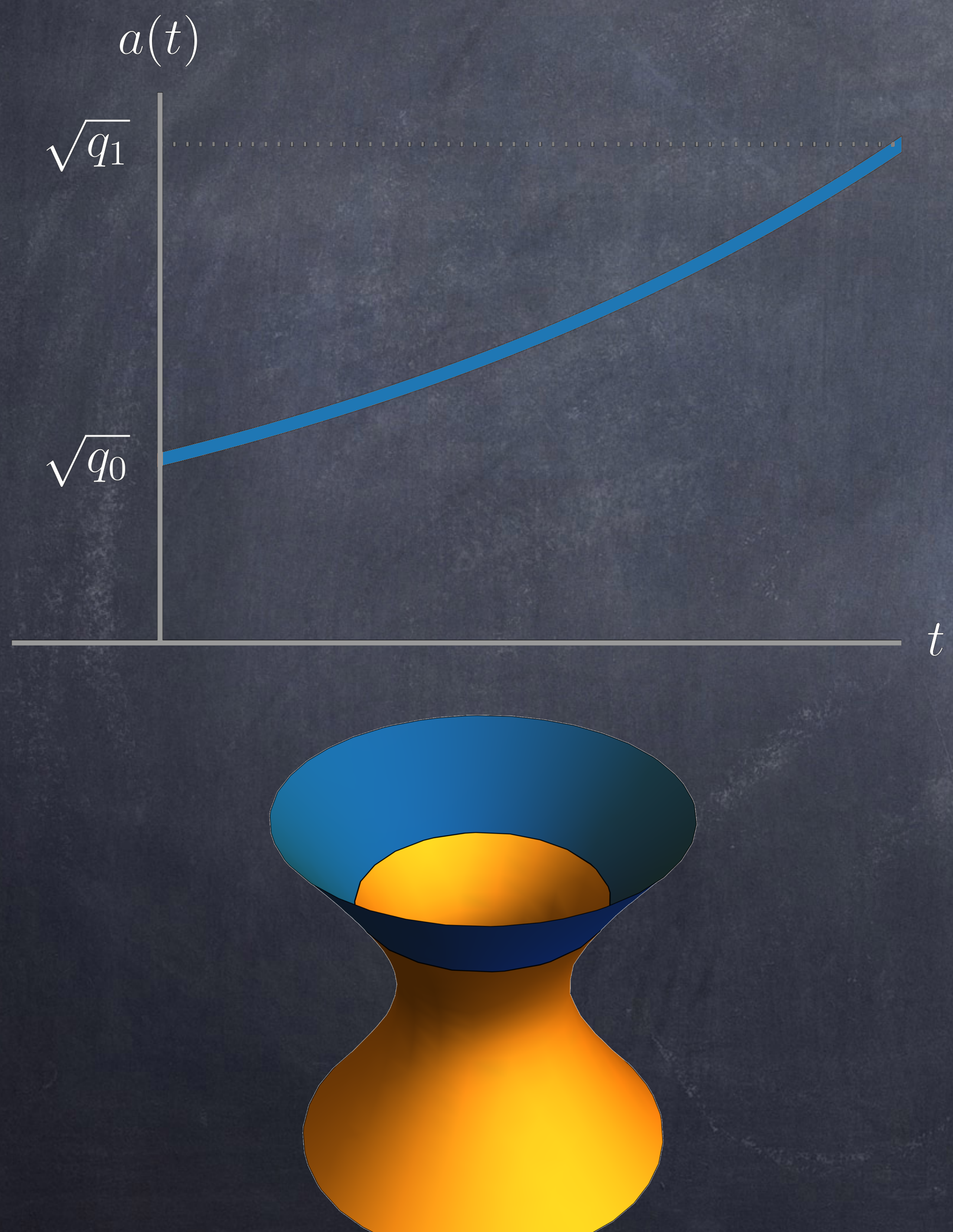
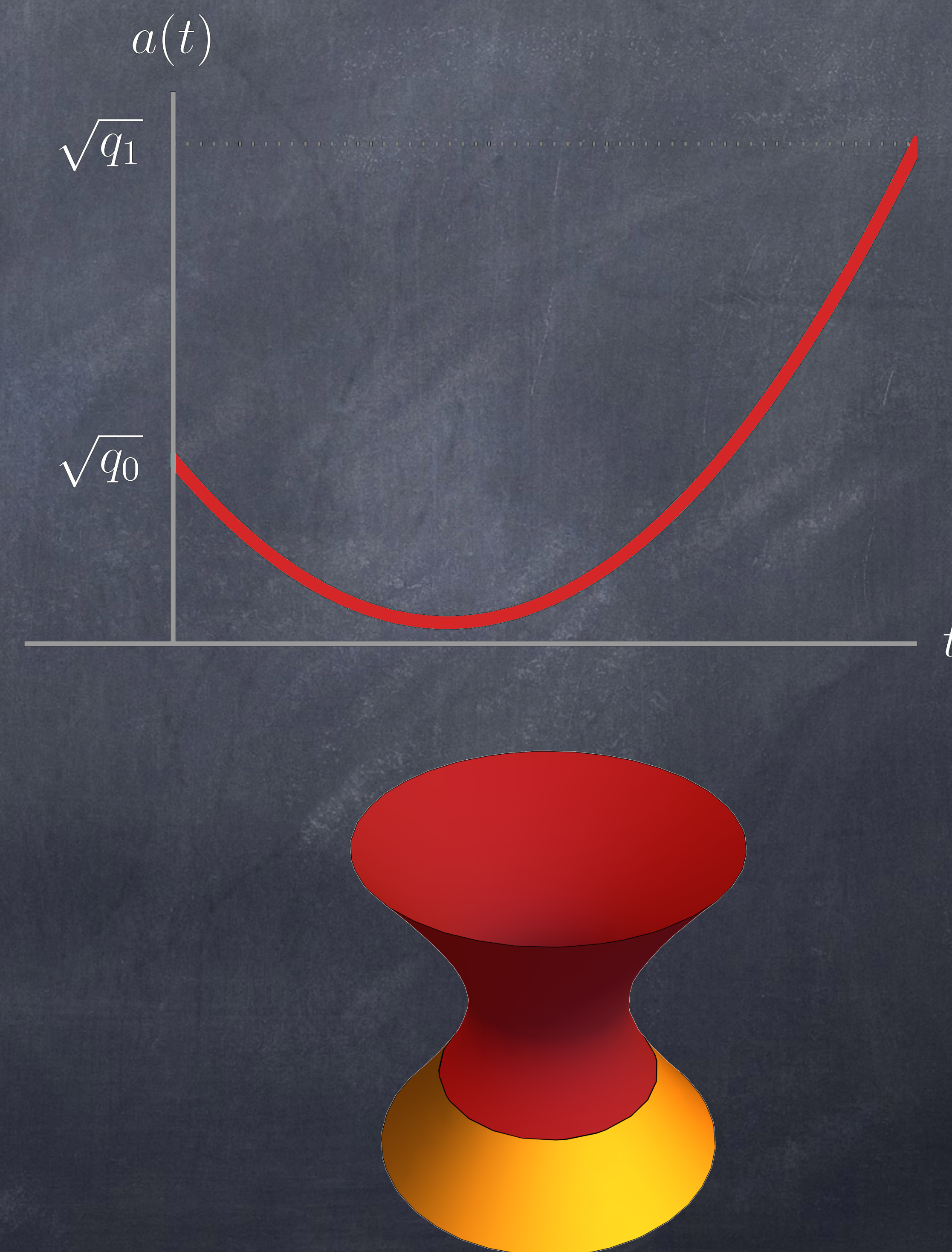
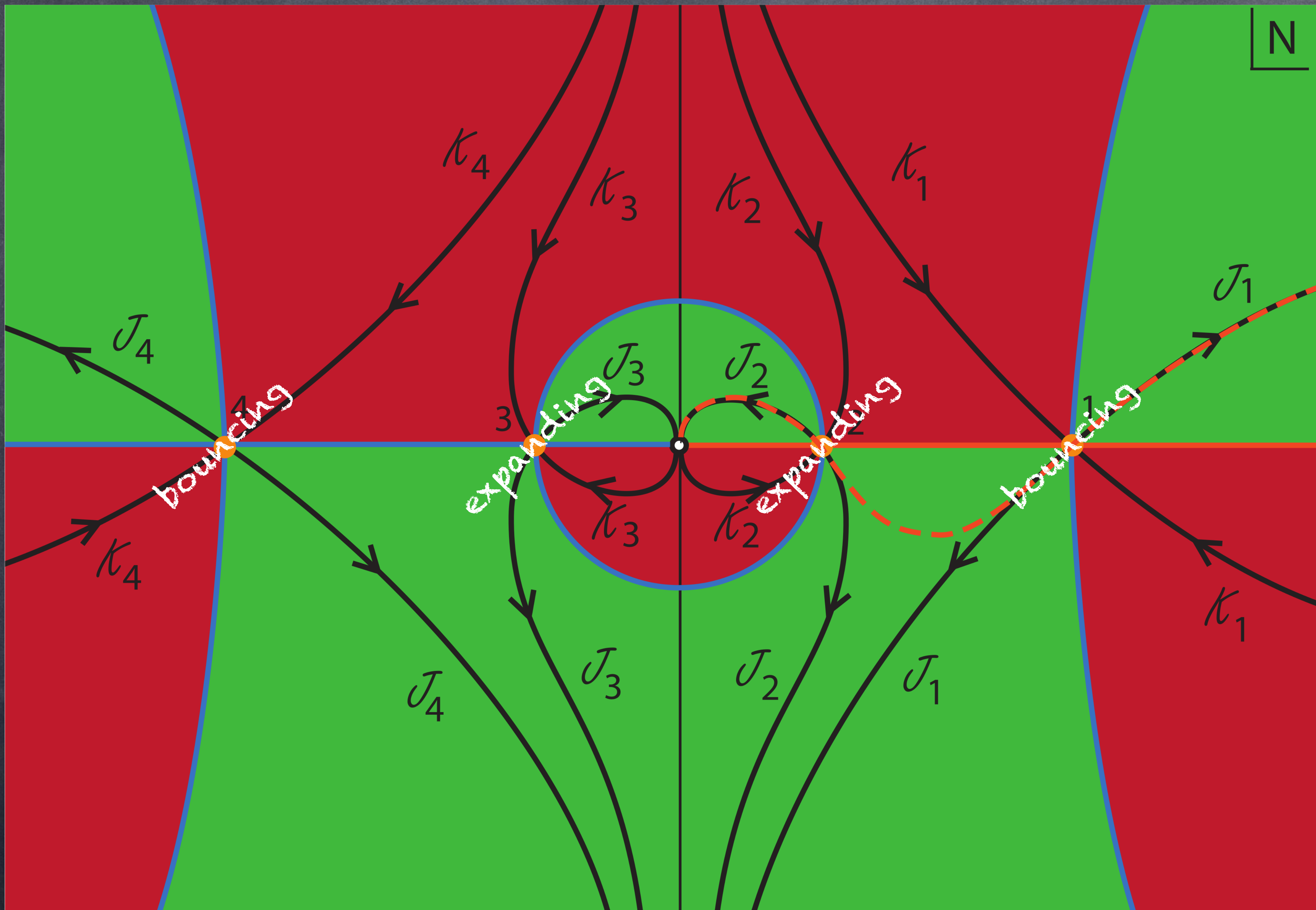
smaller saddle



larger saddle

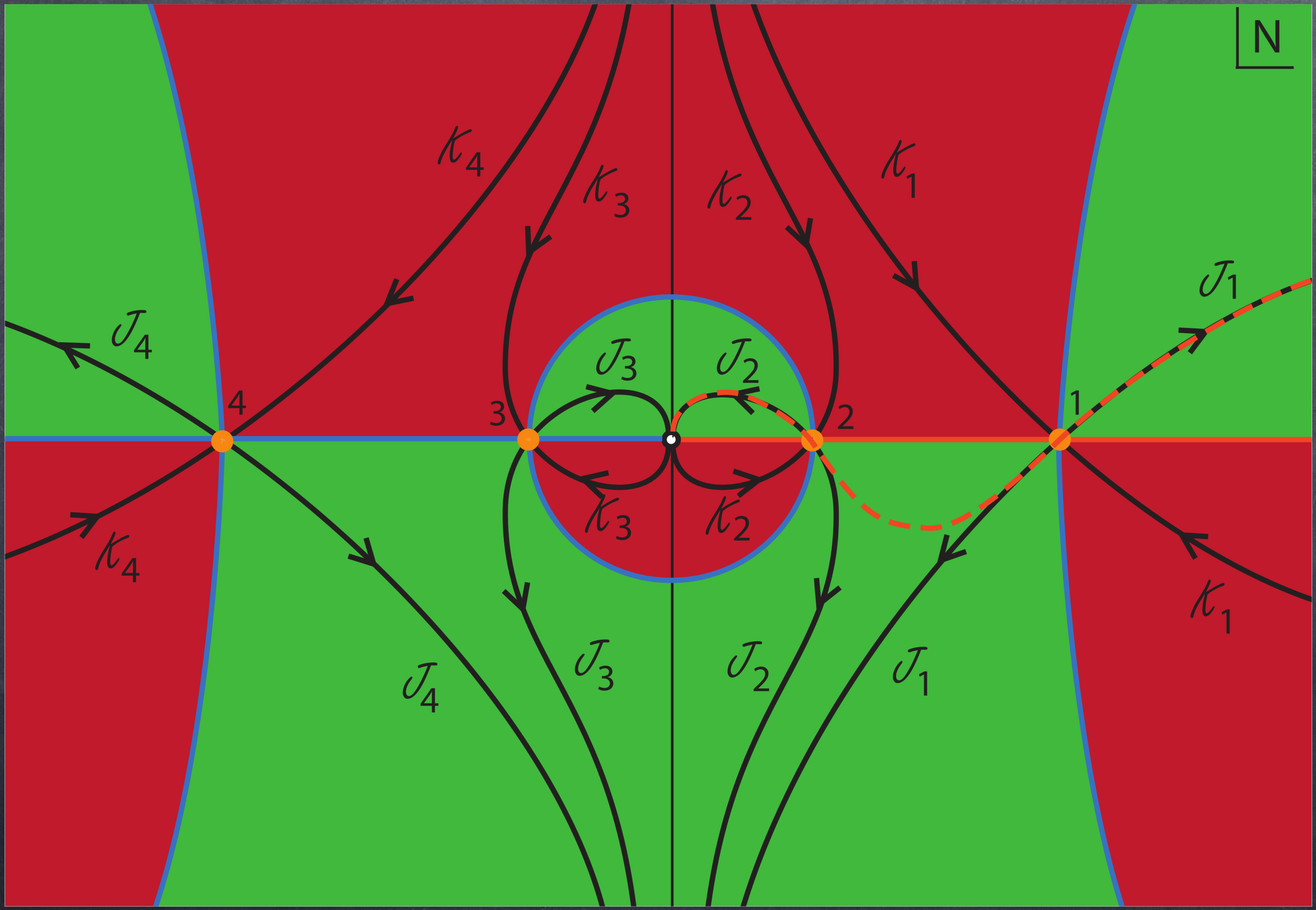


Picard-Lefschetz theory



Feldbrugge-Lehners-Turok [1703.02076]

Picard-Lefschetz theory



Feldbrugge-Lehners-Turok [1703.02076]

Jonas-Lehners-JQ [JHEP08(2022)284]

Kontsevich-Segal to determine allowability:

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\frac{N^2}{q(t)}\mathrm{d}t^2 + q(t)\mathrm{d}\Omega_{(3)}^2 = -\frac{N^2}{q(t(u))}t'(u)^2\mathrm{d}u^2 + q(t(u))\mathrm{d}\Omega_{(3)}^2$$

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$$\Sigma(u) = \left|\mathrm{Arg}\left[-\frac{N^2}{q(t(u))}t'(u)^2\right]\right| + 3\left|\mathrm{Arg}\left[q(t(u))\right]\right| < \pi \quad \forall\, u$$

Jonas-Lehners-JQ [JHEP08(2022)284]

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$$\Rightarrow \quad F_{\min}(q(t(u)),N) < \mathrm{Arg}\left(t'(u)\right) < F_{\max}(q(t(u)),N)$$

Jonas-Lehners-JQ [JHEP08(2022)284]

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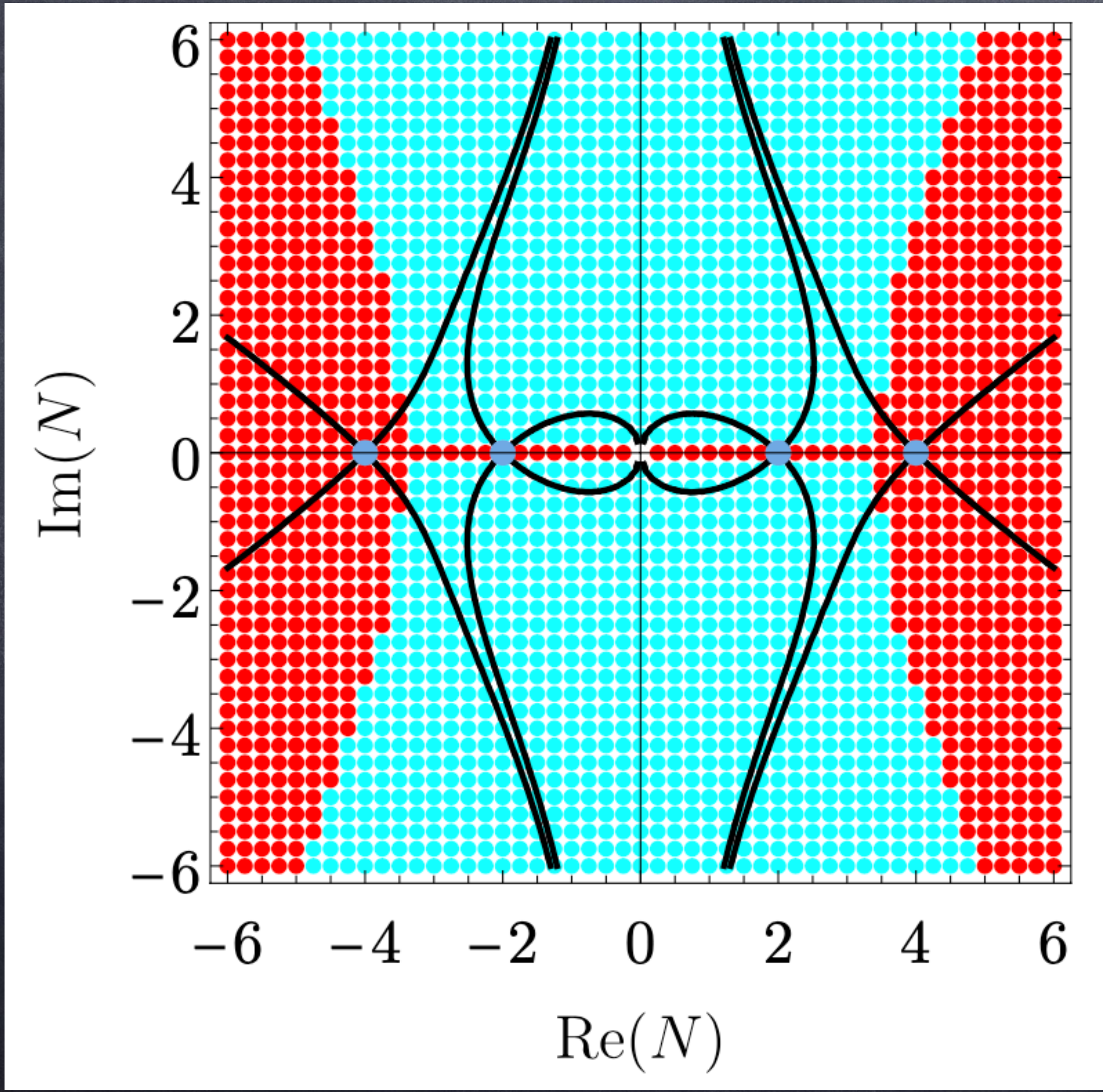
$$\Rightarrow \; F_{\mathrm{min}}(q(t(u)),N) < \mathrm{Arg}\left(t'(u)\right) < F_{\mathrm{max}}(q(t(u)),N)$$

$$\Rightarrow \; t_{\mathrm{min}}(u) < t(u) < t_{\mathrm{max}}(u)$$

de Sitter classical transitions with $\Lambda = 3,\ q_0 = 2,\ q_1 = 10$ so $N_{\rm SP} \in \{\pm 2, \pm 4\}$



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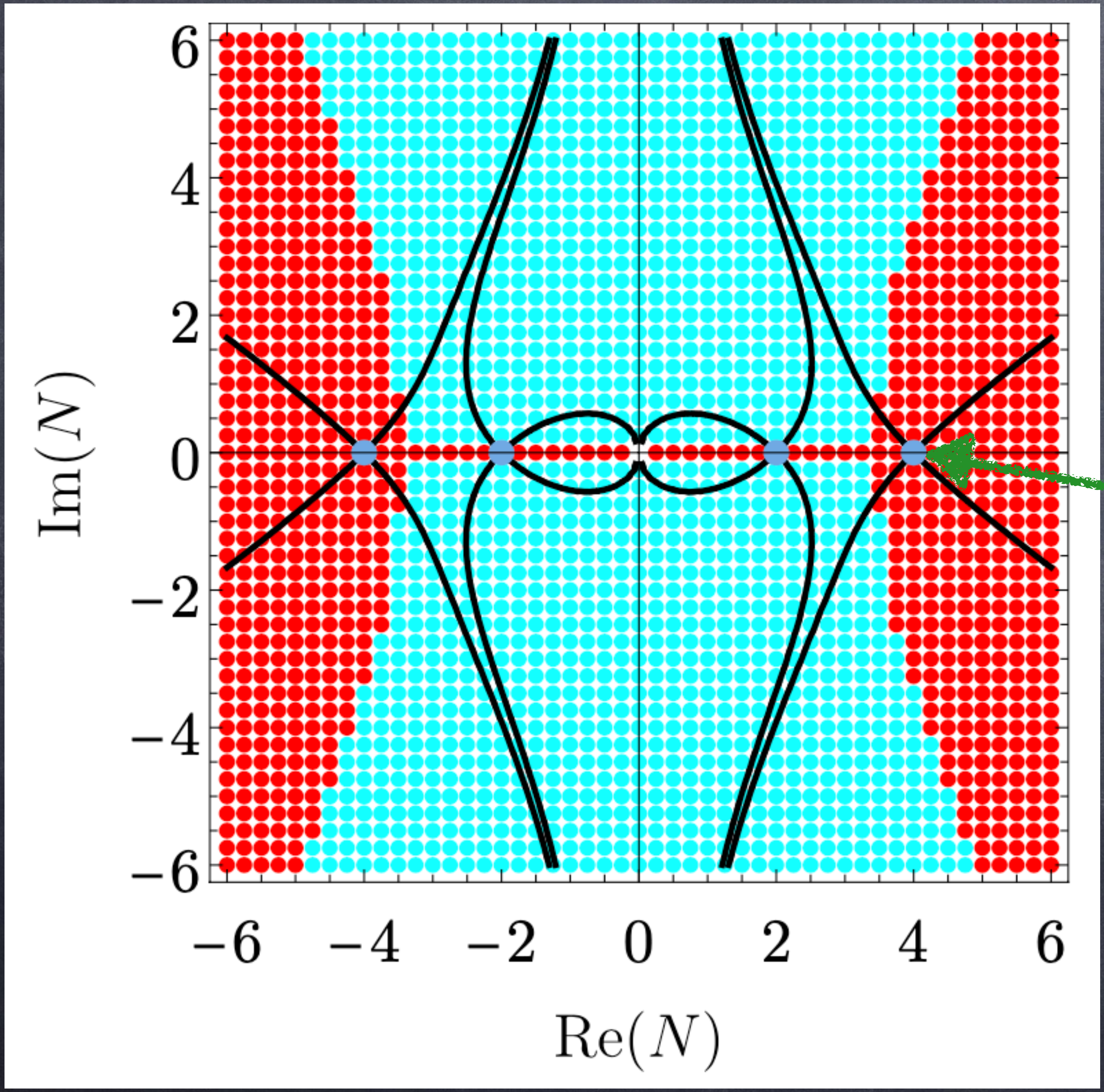
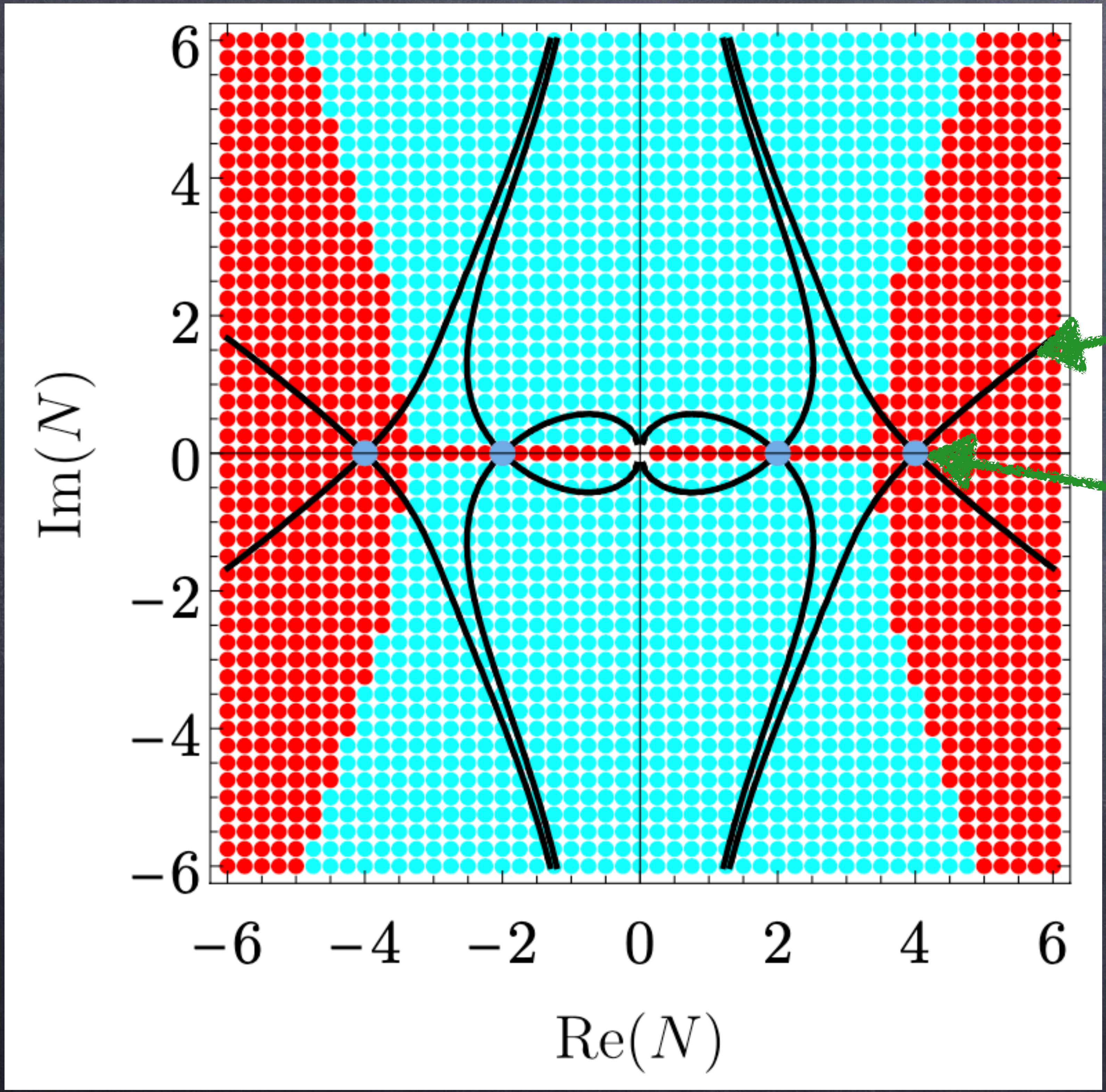
de Sitter classical transitions with $\Lambda = 3$, $q_0 = 2$, $q_1 = 10$ so $N_{\rm SP} \in \{\pm 2, \pm 4\}$



saddle points

Jonas-Lehners-JQ [JHEP08(2022)284]

de Sitter classical transitions with $\Lambda = 3$, $q_0 = 2$, $q_1 = 10$ so $N_{\text{SP}} \in \{\pm 2, \pm 4\}$

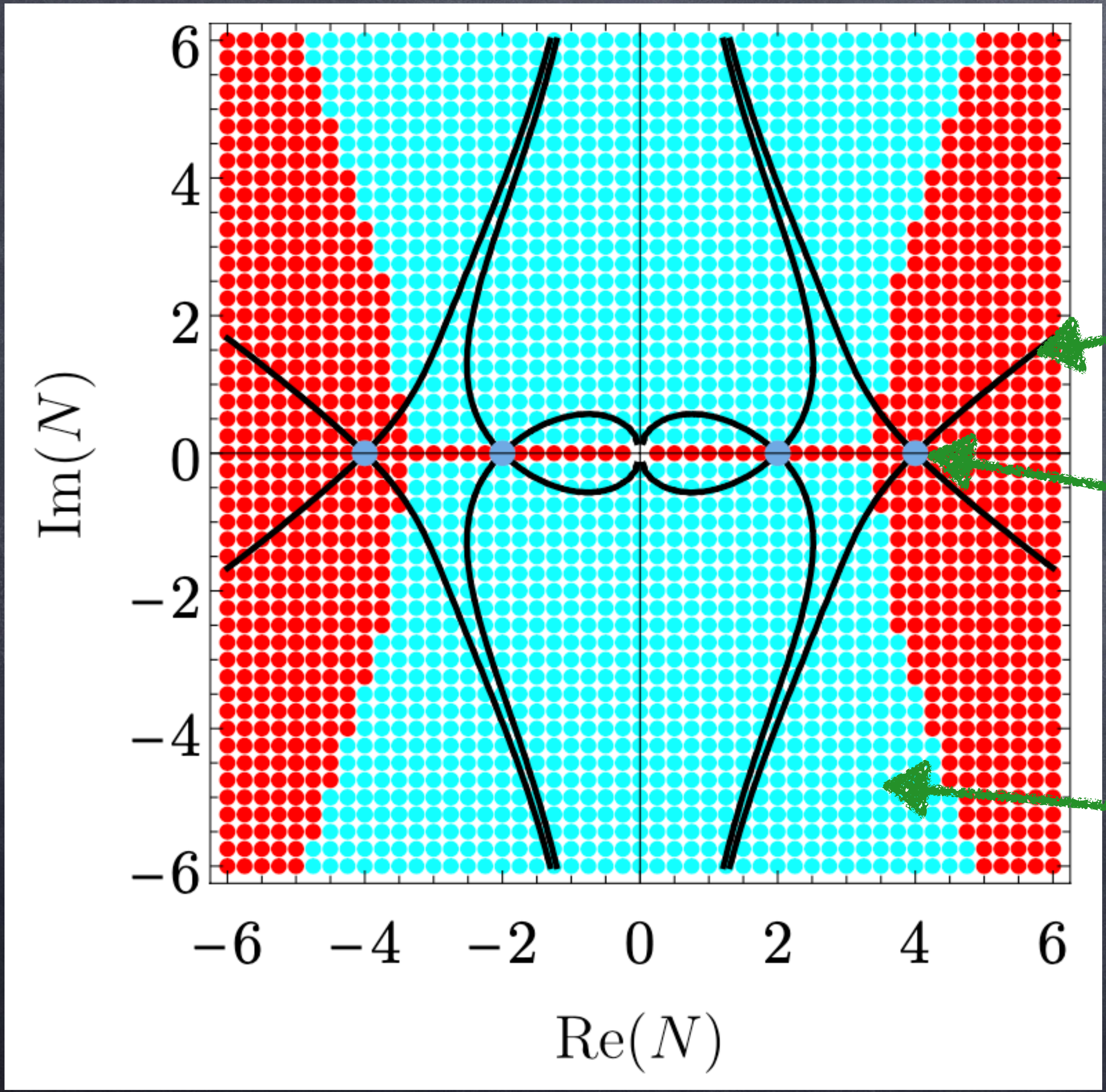


steepest descent contours

saddle points

Jonas-Lehners-JQ [JHEP08(2022)284]

de Sitter classical transitions with $\Lambda = 3$, $q_0 = 2$, $q_1 = 10$ so $N_{SP} \in \{\pm 2, \pm 4\}$



steepest descent contours

saddle points

allowable

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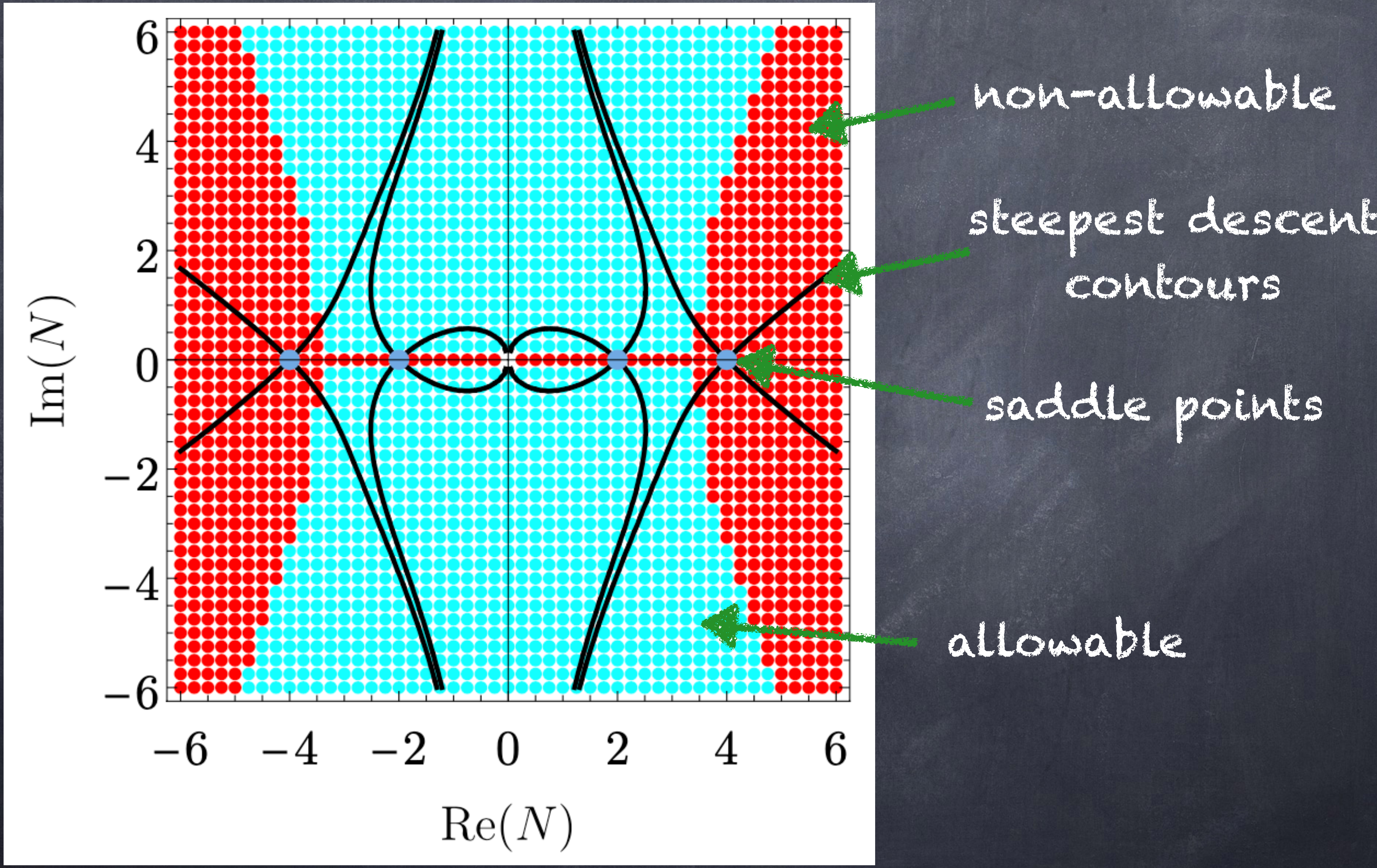

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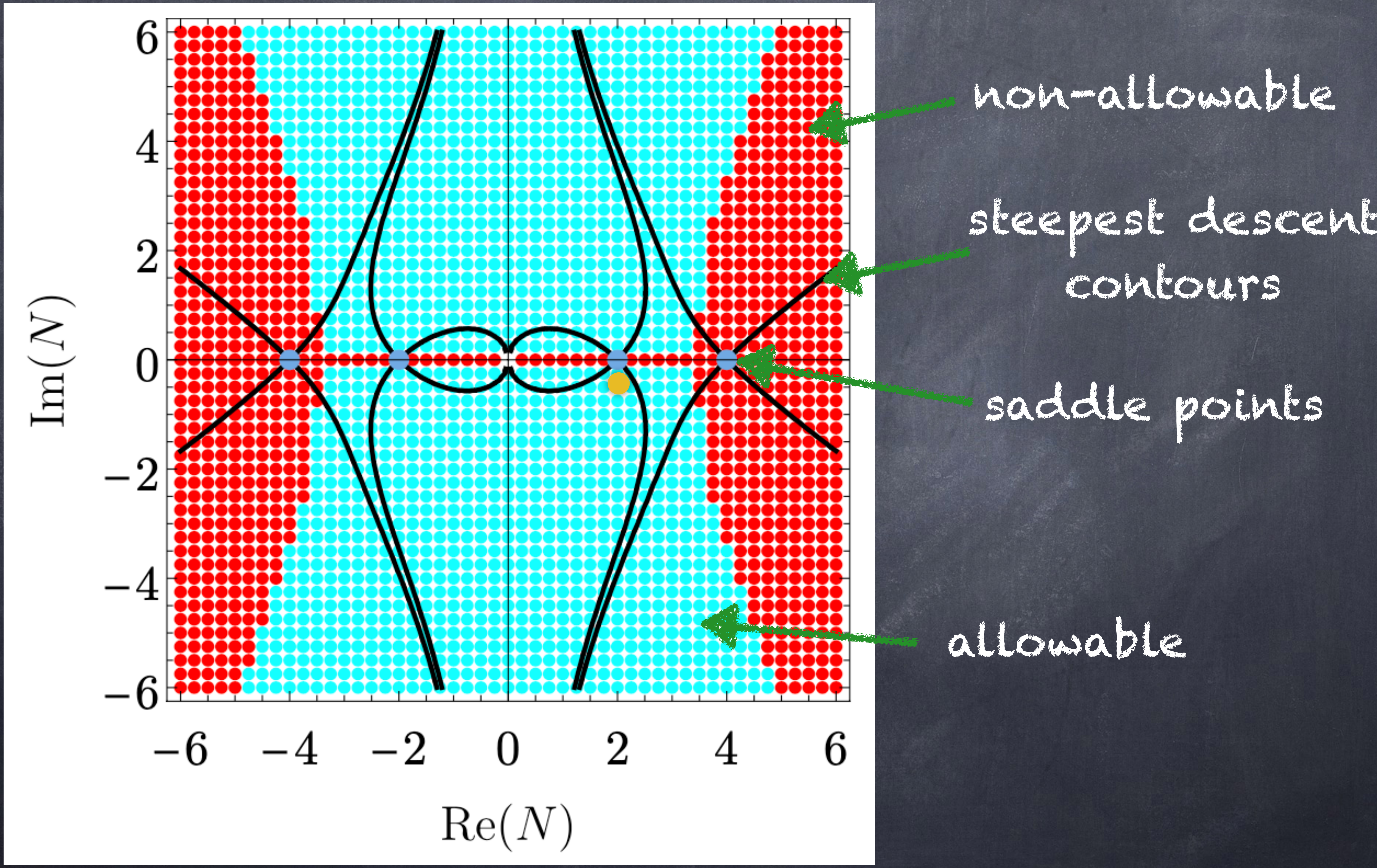

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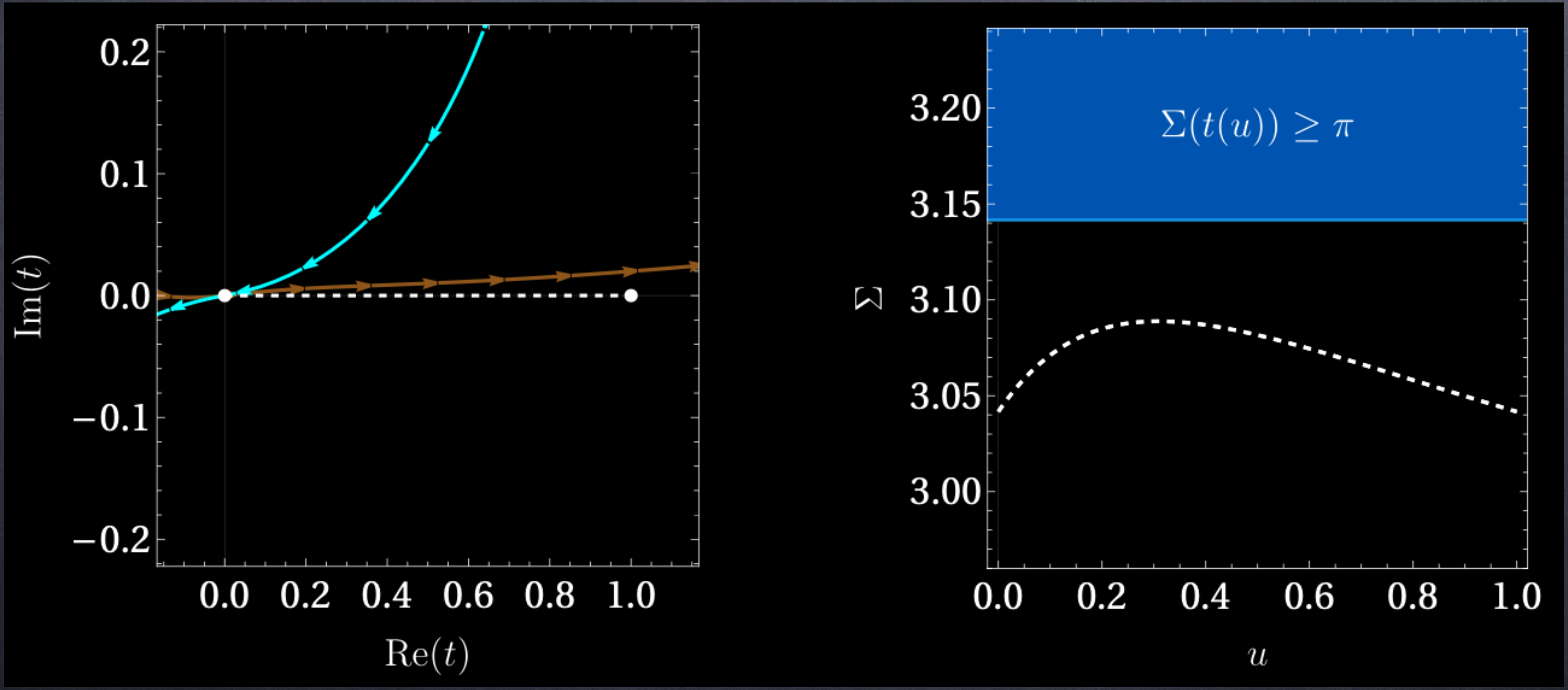
$$N = 2 - i/10$$



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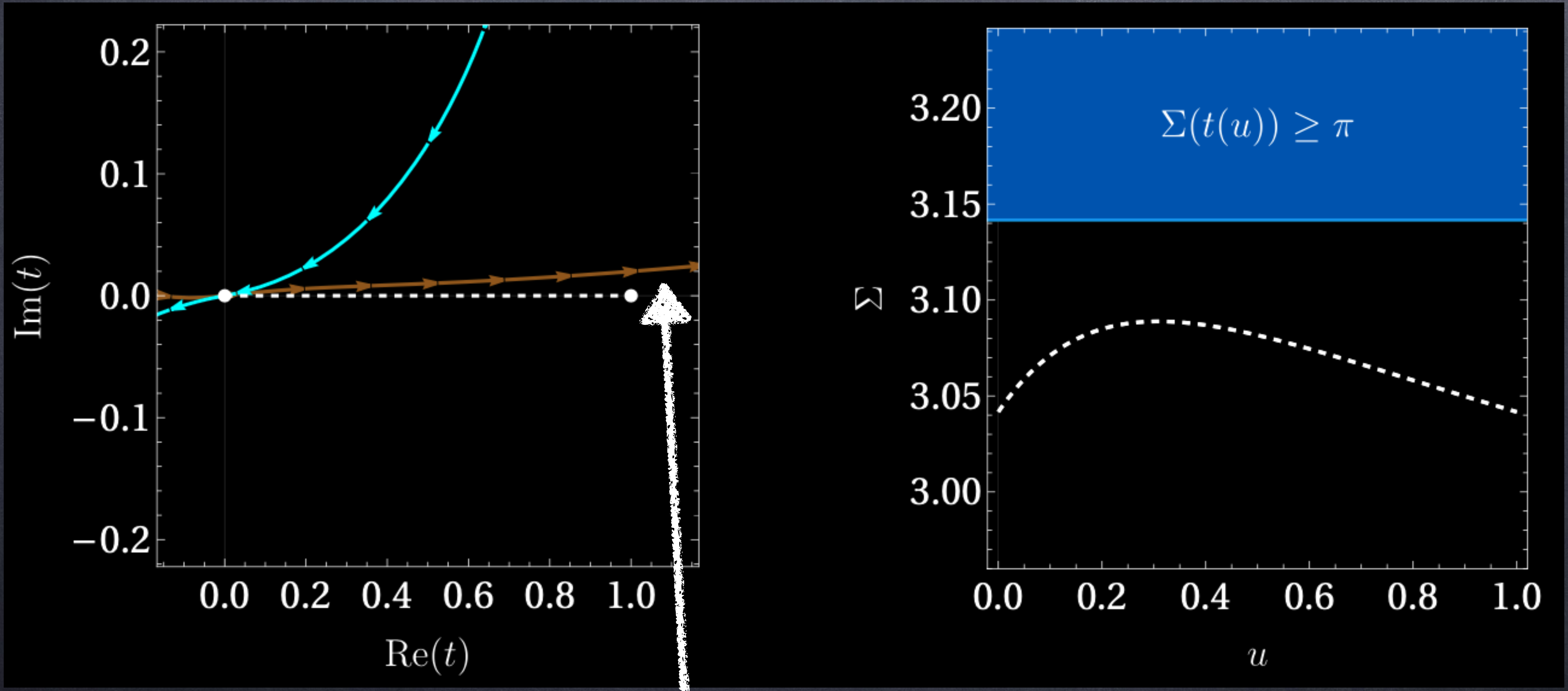

maximal angle

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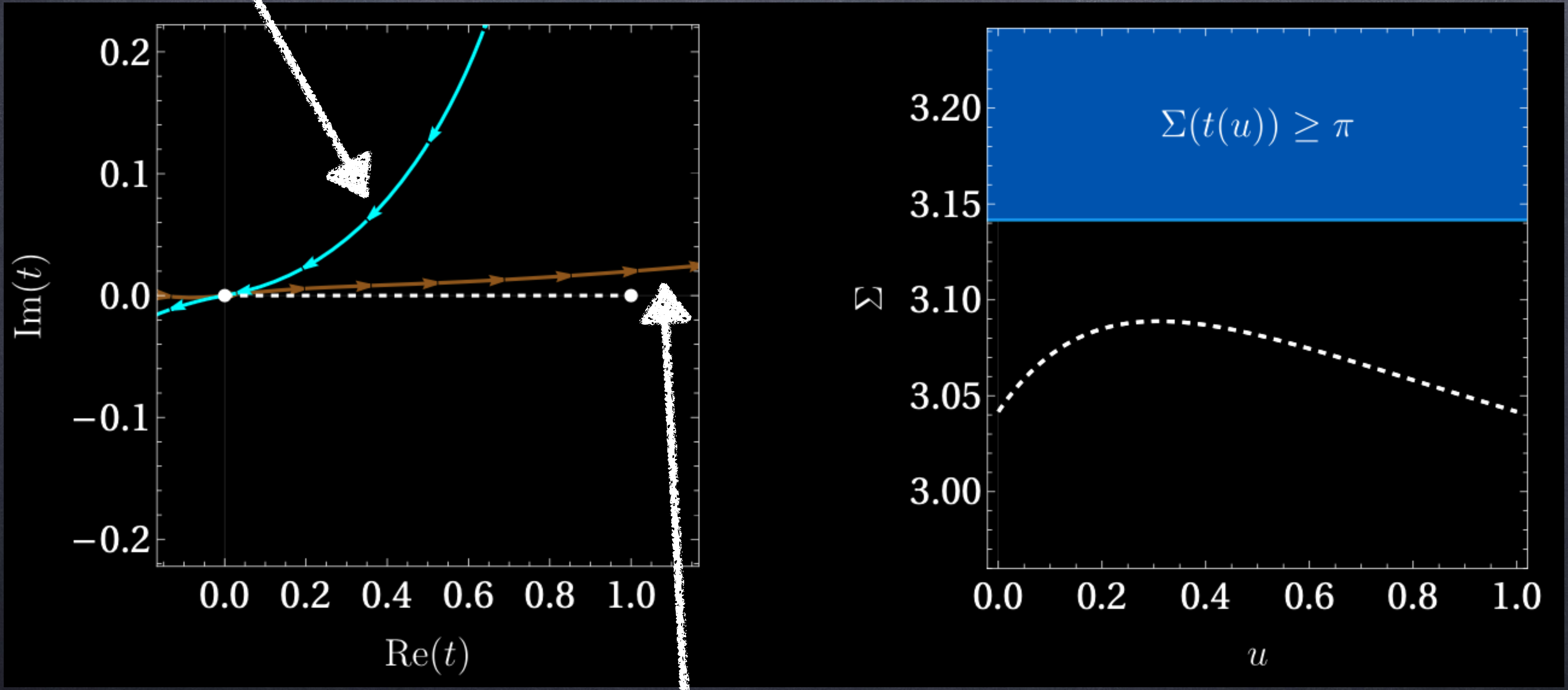

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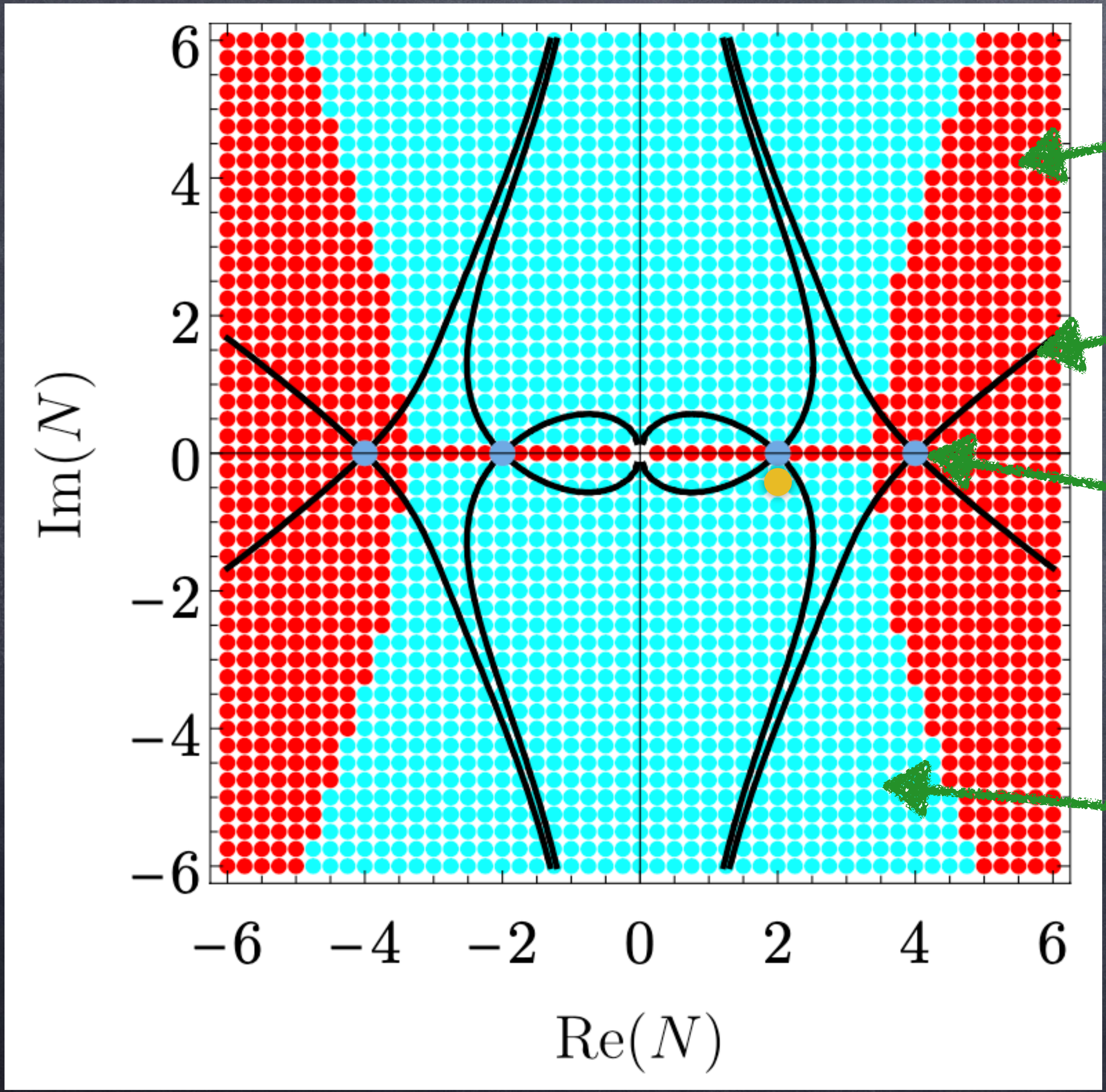

non-allowable

steepest descent contours

saddle points

allowable

Jonas-Lehners-JQ [JHEP08(2022)284]

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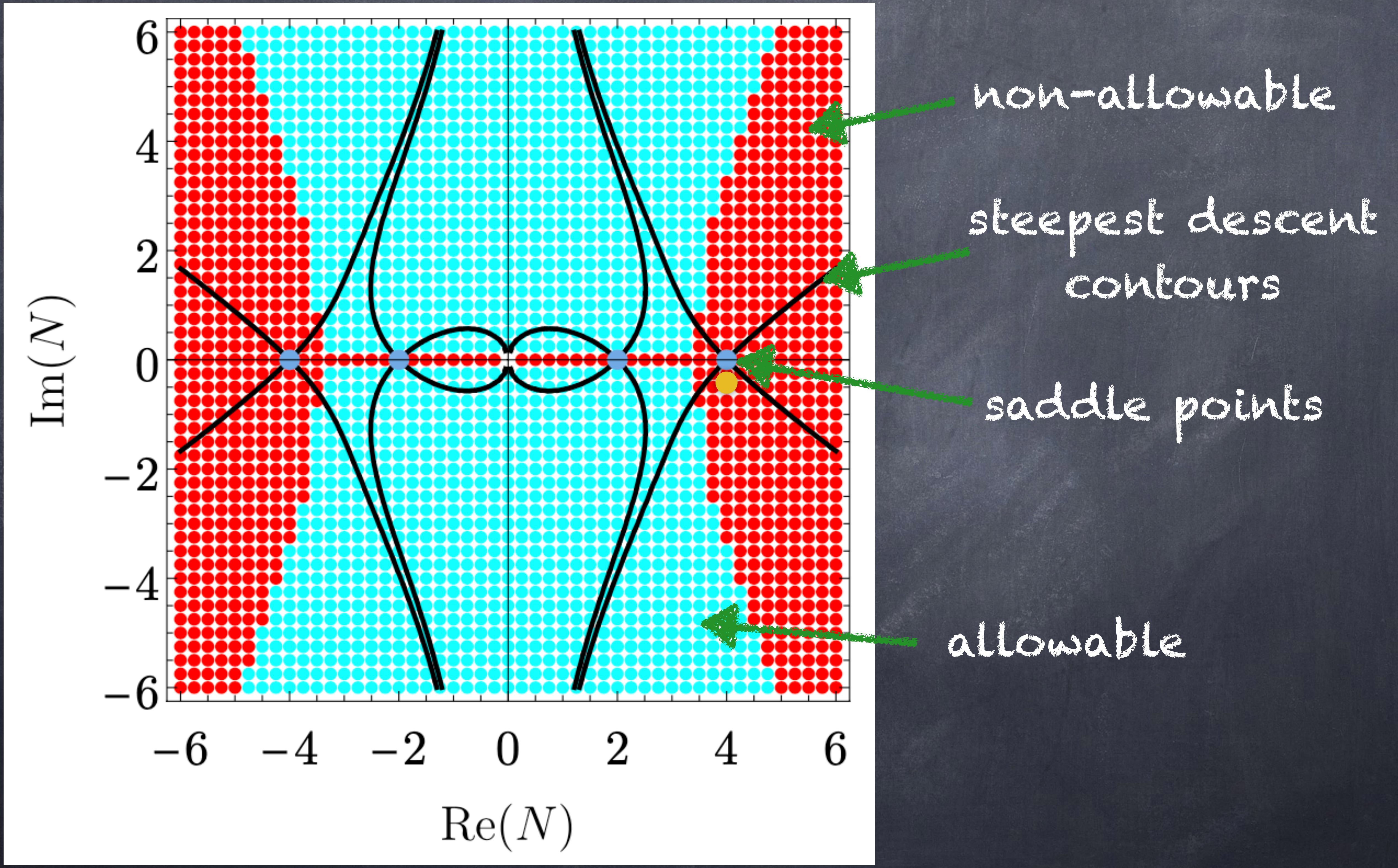

Jonas-Lehners-JQ [JHEP08(2022)284]

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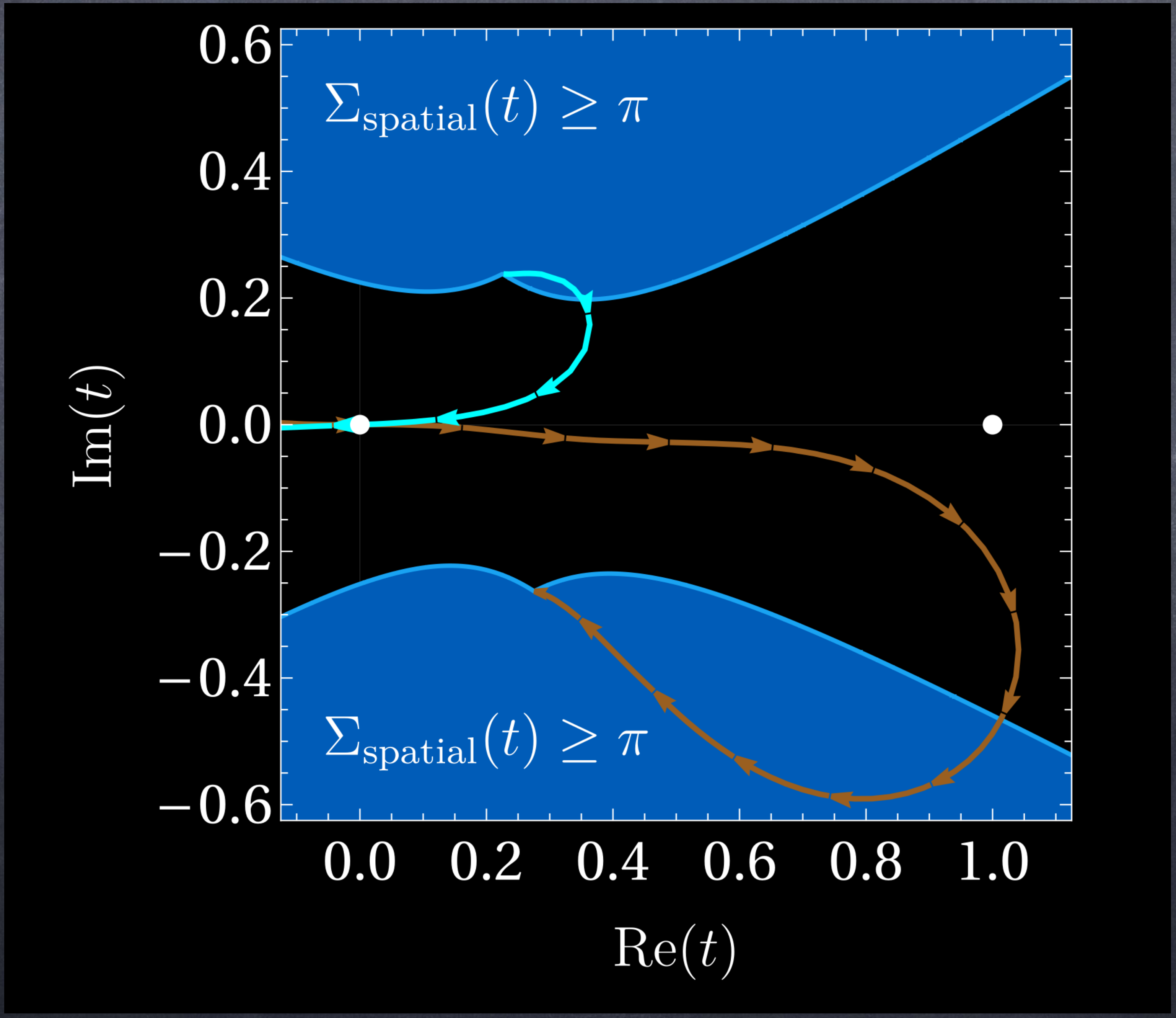

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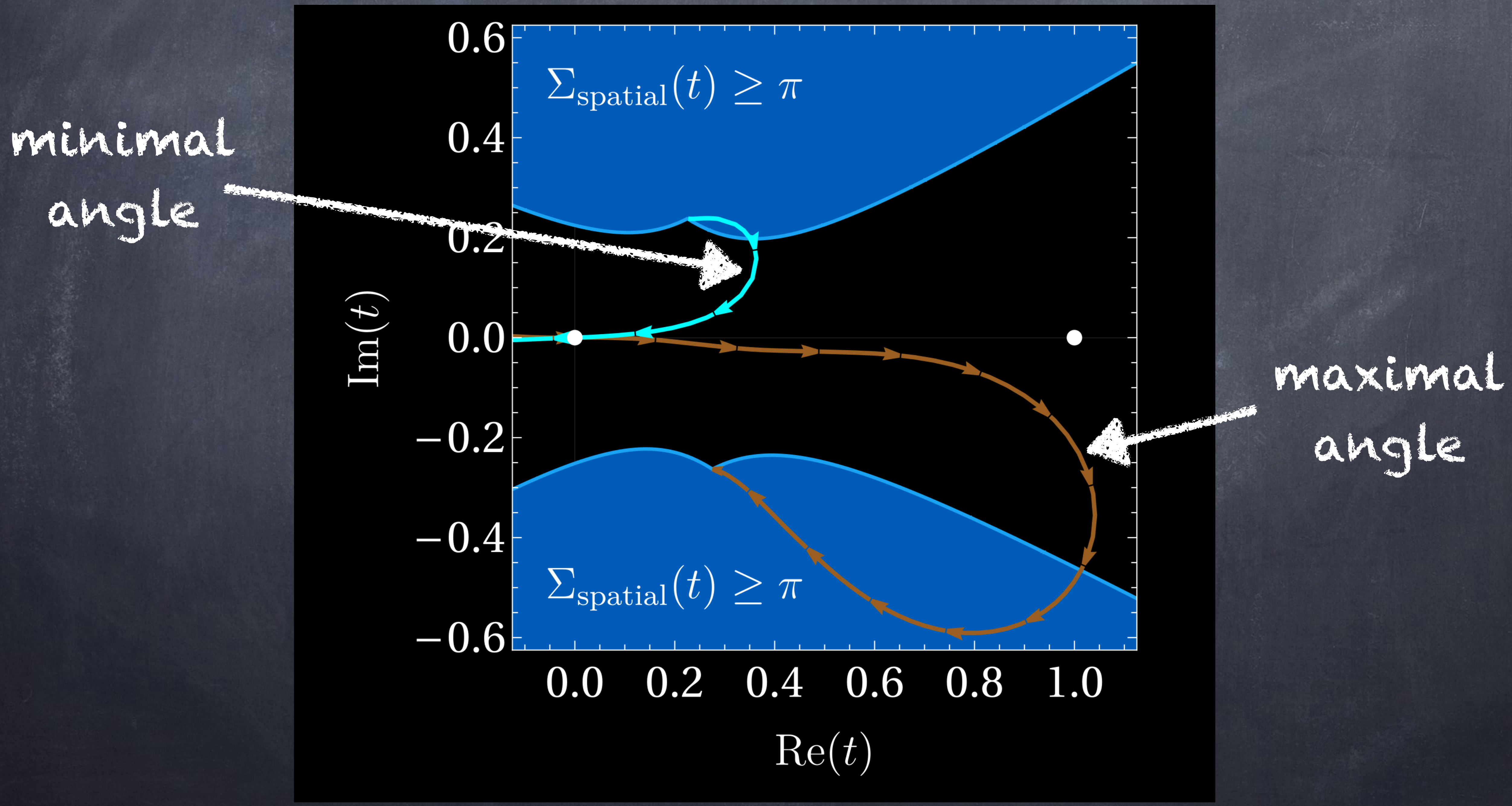
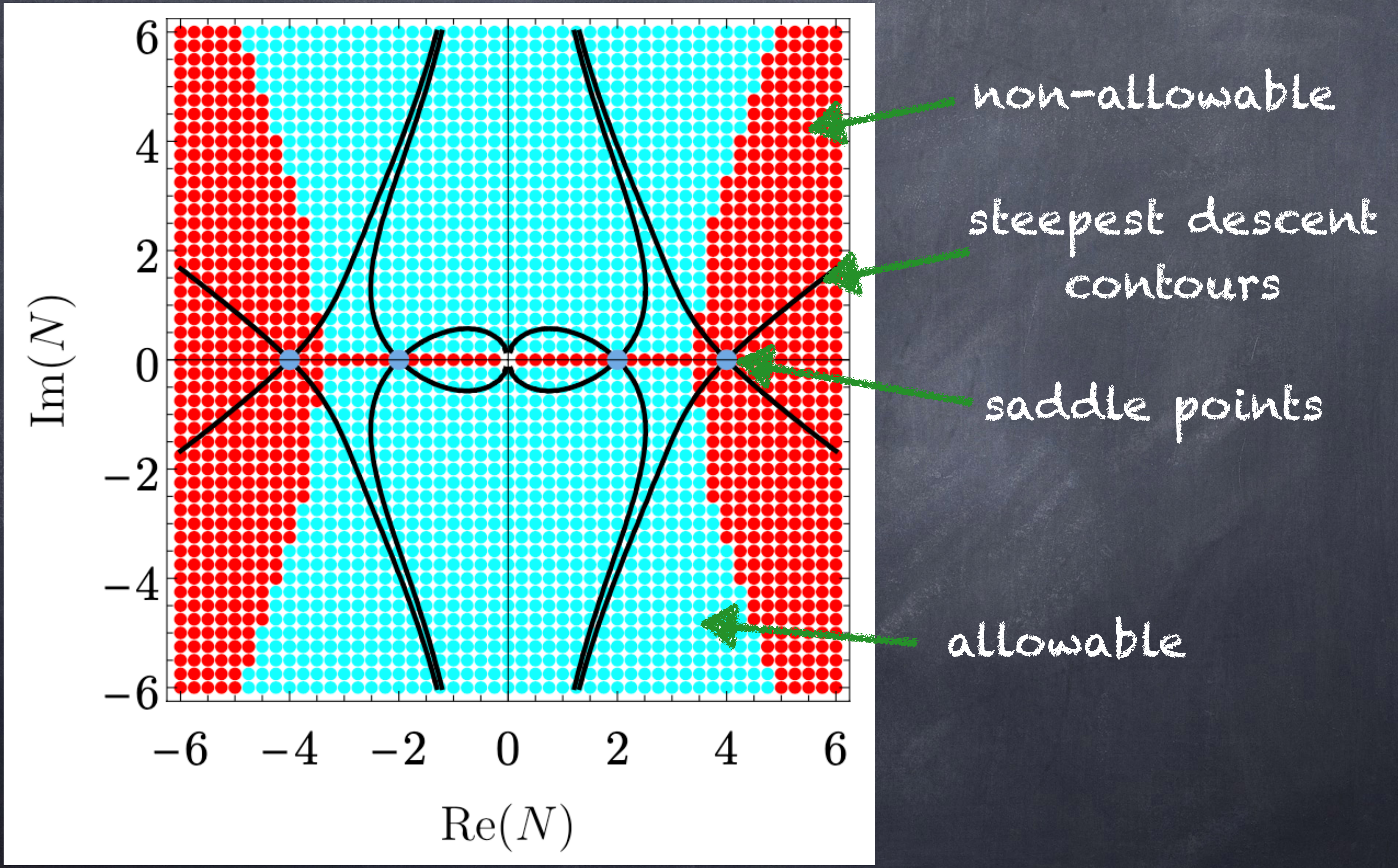


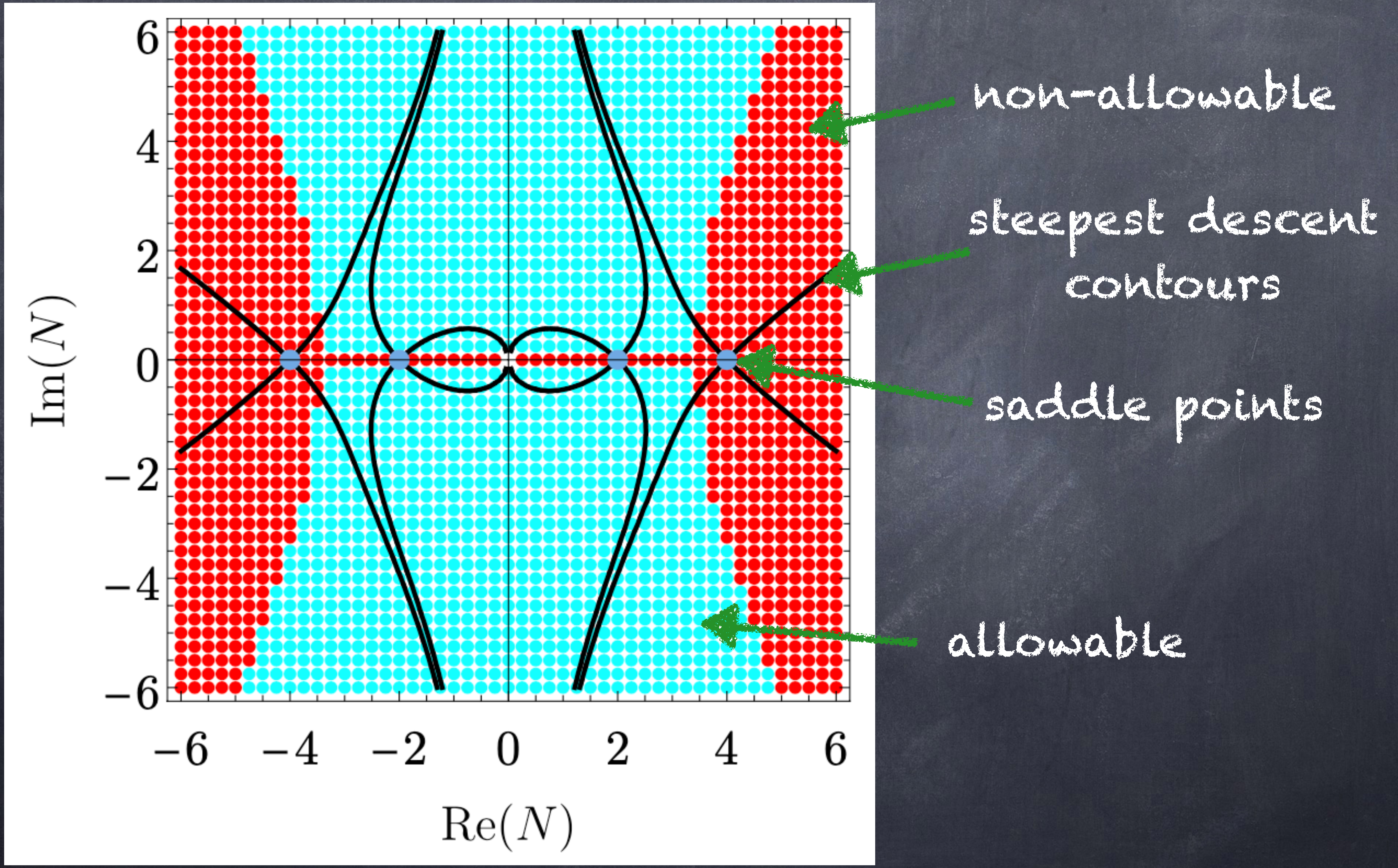
Jonas-Lehners-JQ [JHEP08(2022)284]

- Bouncing saddles are unreachable



Jonas-Lehners-JQ [JHEP08(2022)284]

- Bouncing saddles are unreachable
- Lefschetz thimbles are cut off



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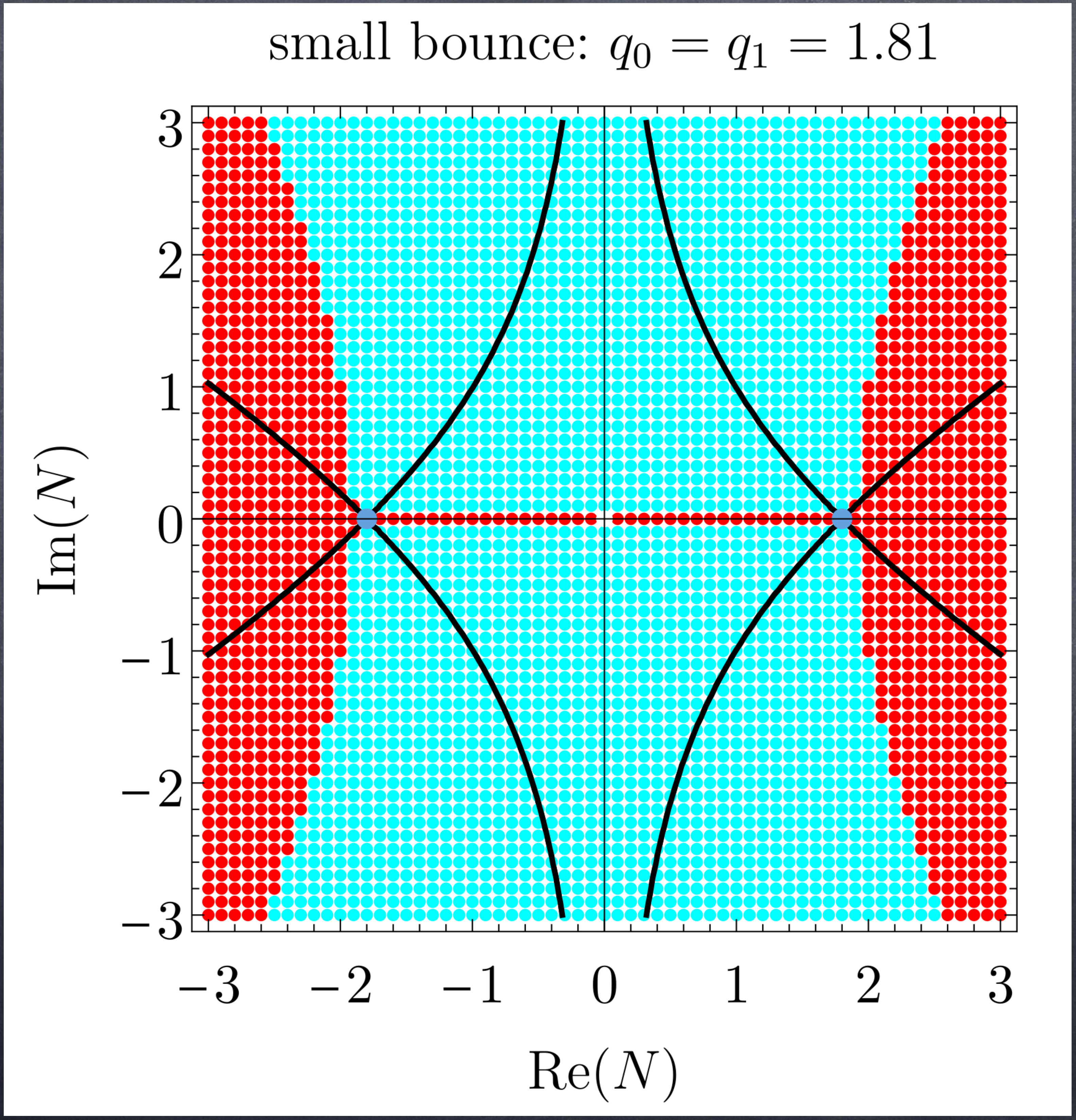
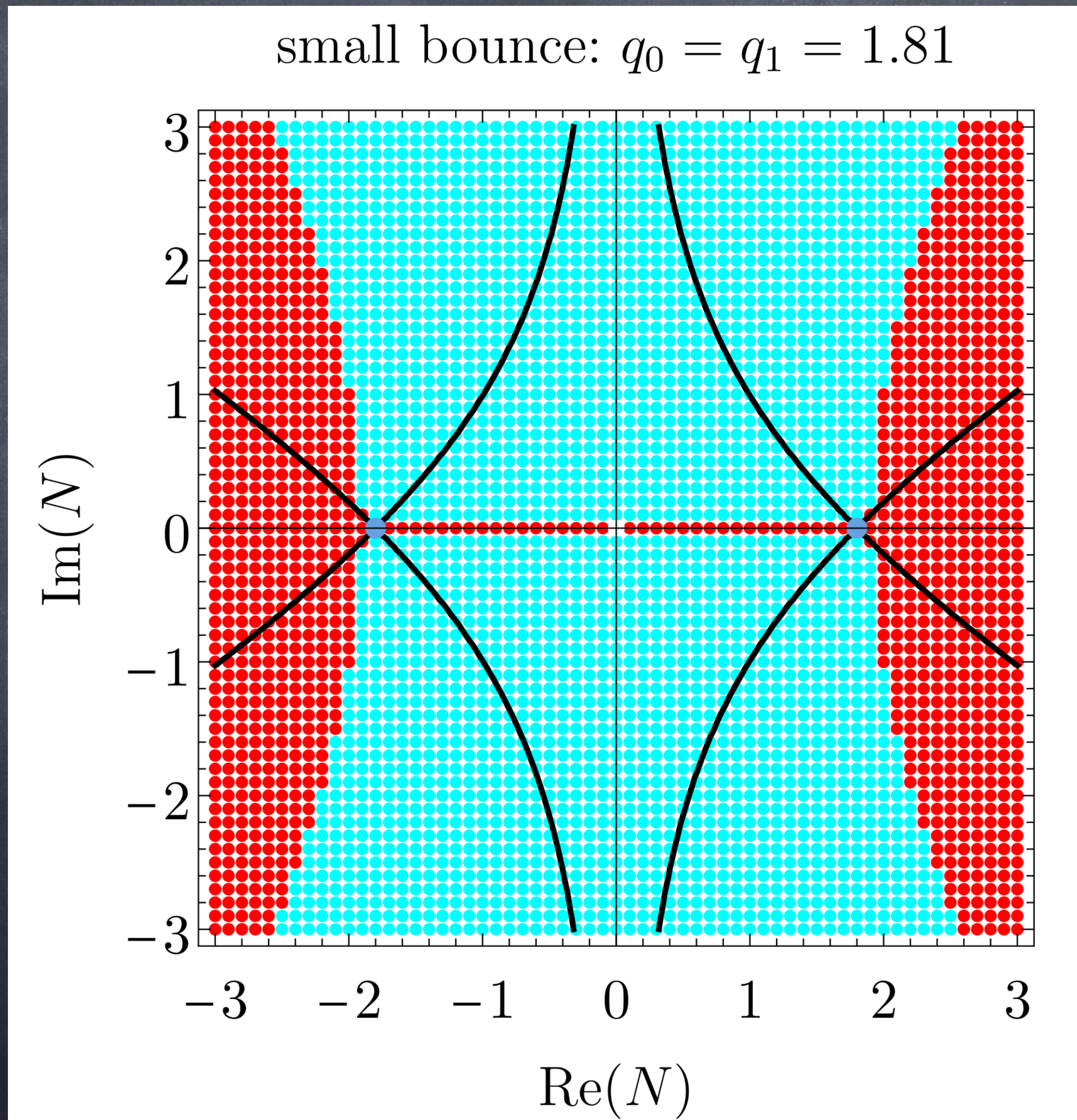


- Bouncing saddles are unreachable

  → perhaps only true for "large" bounces



What goes wrong around the (large) bouncing saddle point?

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Consider a real massive scalar: $\displaystyle\int \mathscr{D}\phi\, e^{\frac{i}{\hbar}\int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}m^2\phi^2\right)}$

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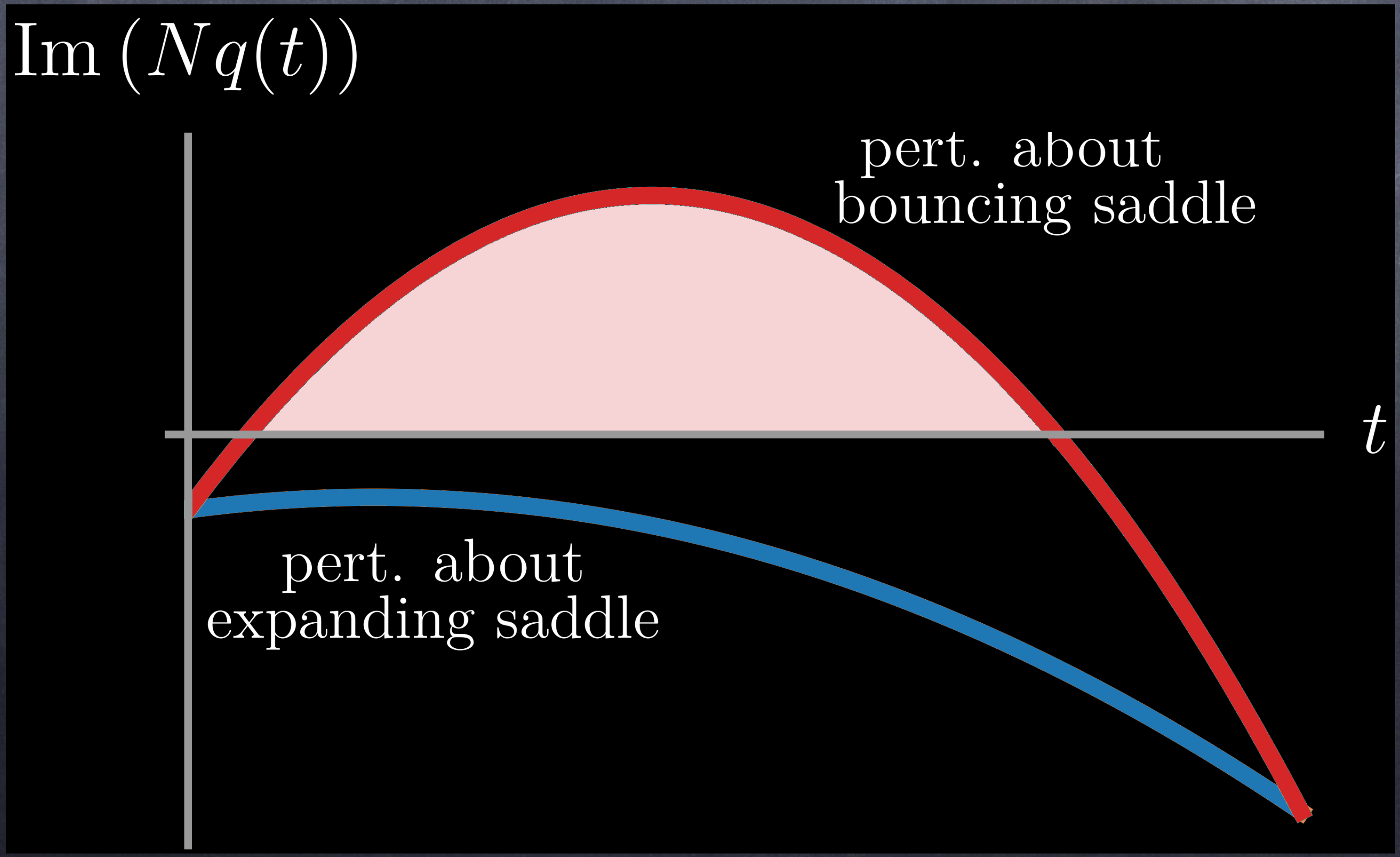

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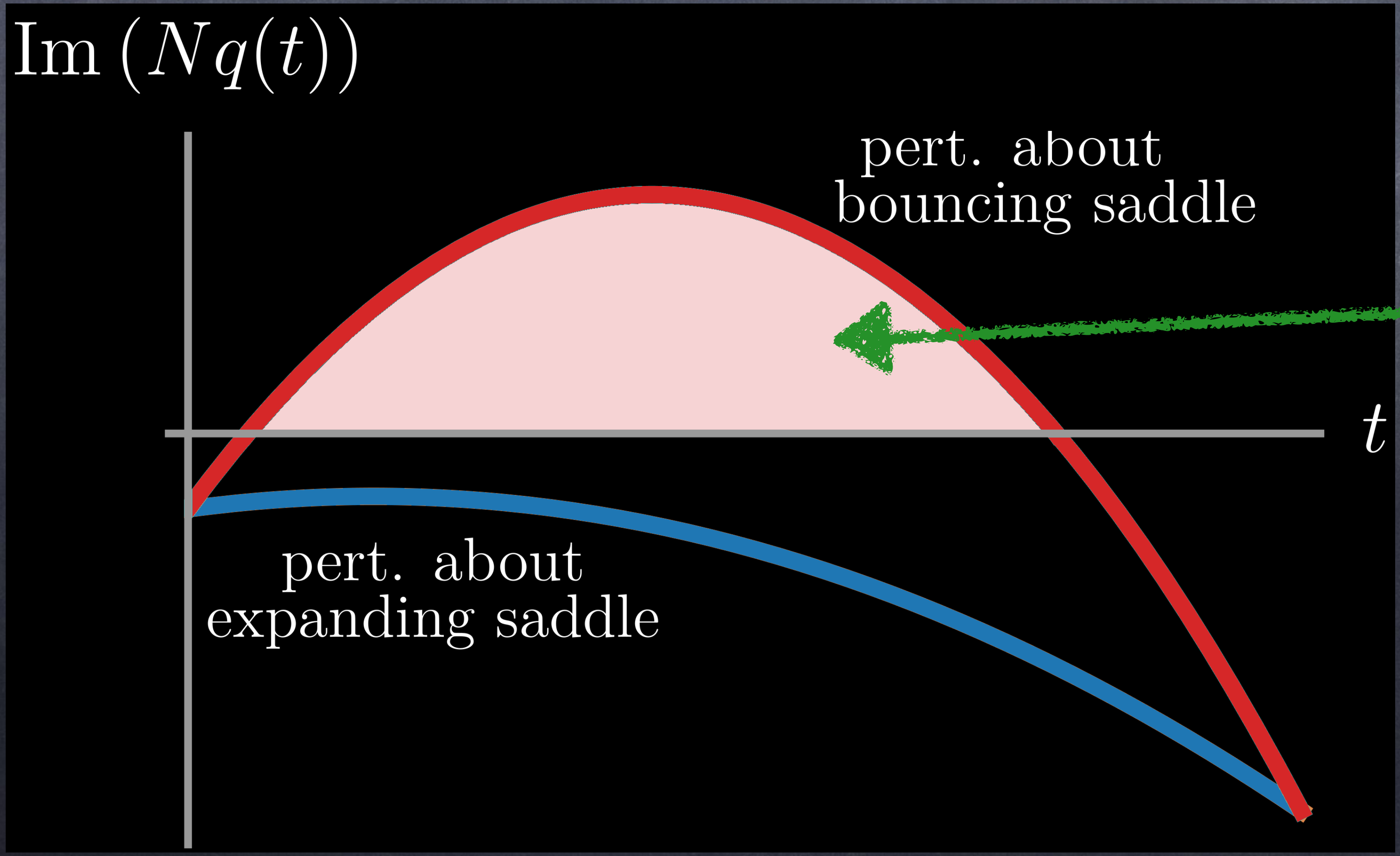
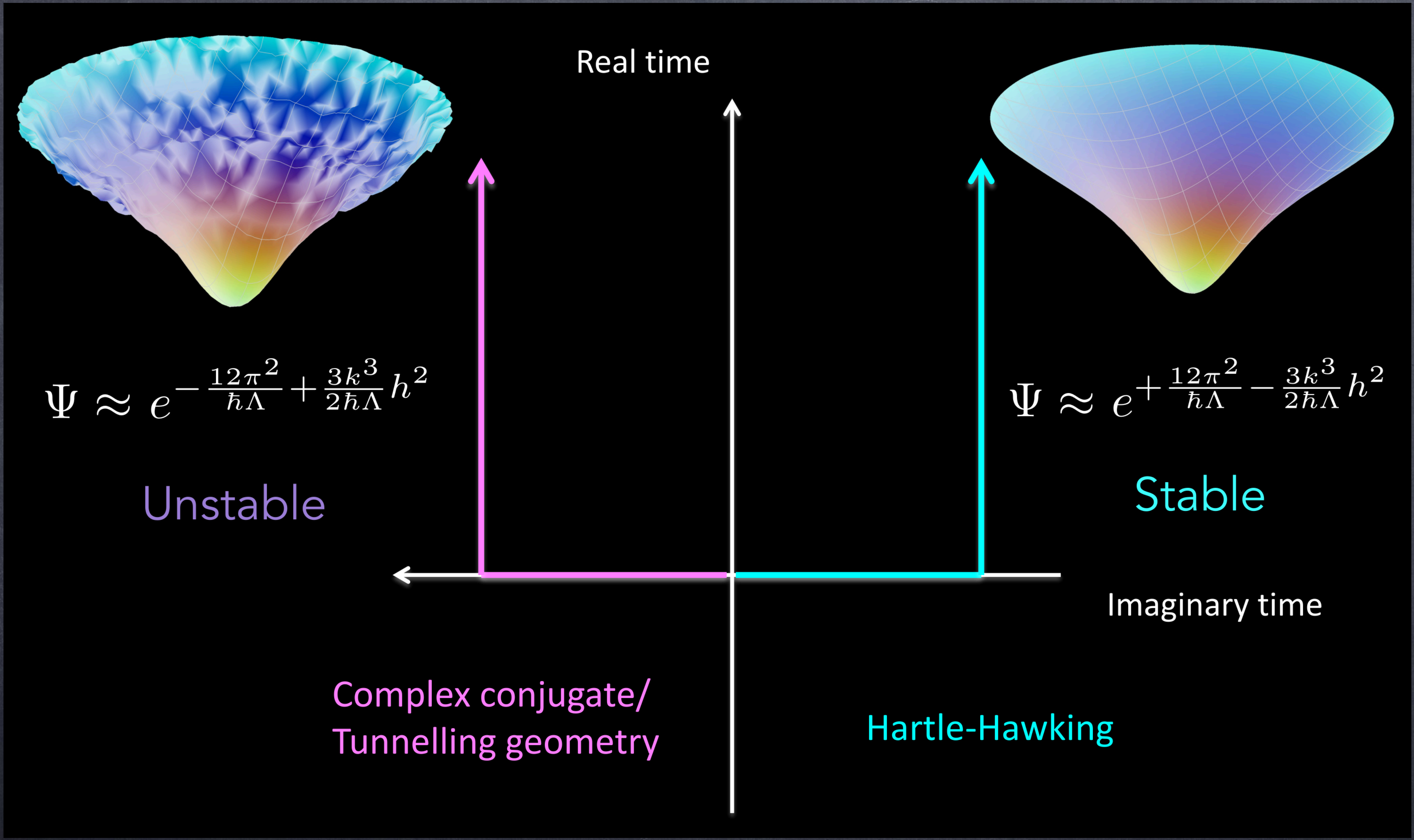
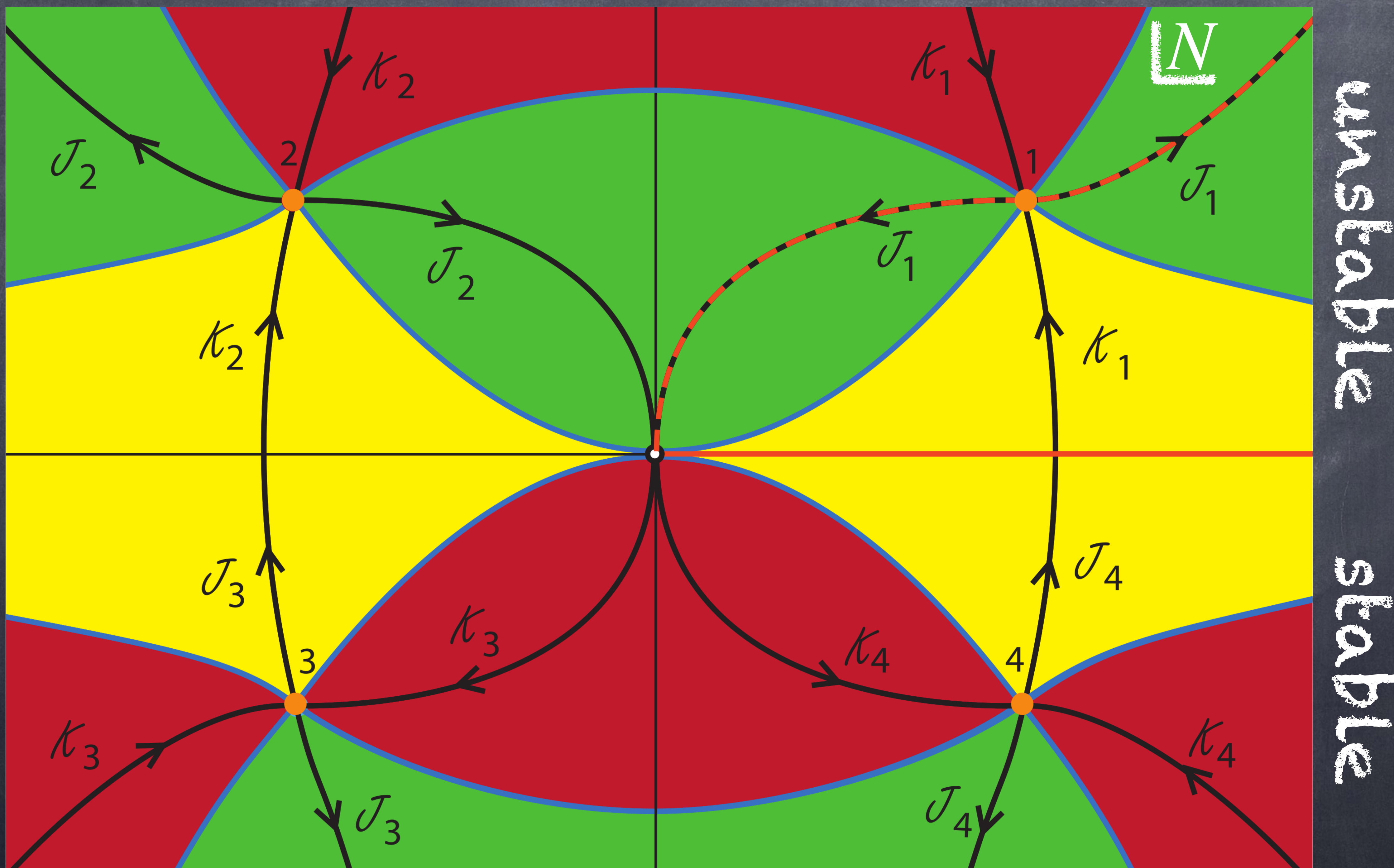


instability

Back to the no-boundary proposal



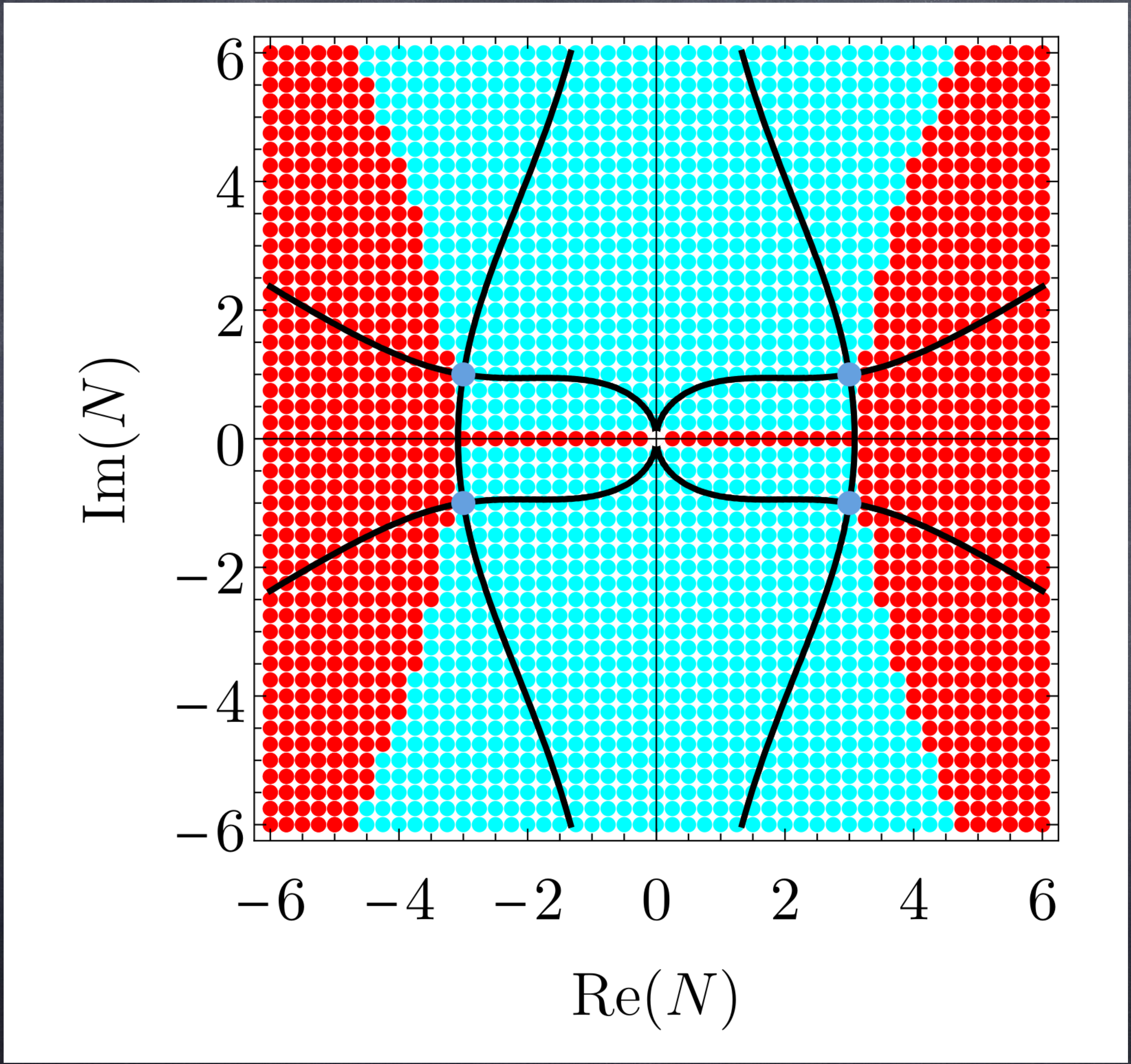
Picard-Lefschetz picks up the unstable saddle



Feldbrugge-Lehners-Turok [1703.02076]

Jonas-Lehners-JQ [JHEP08(2022)284]

$\Lambda = 3,\ q_0 = 0,\ q_1 = 10$



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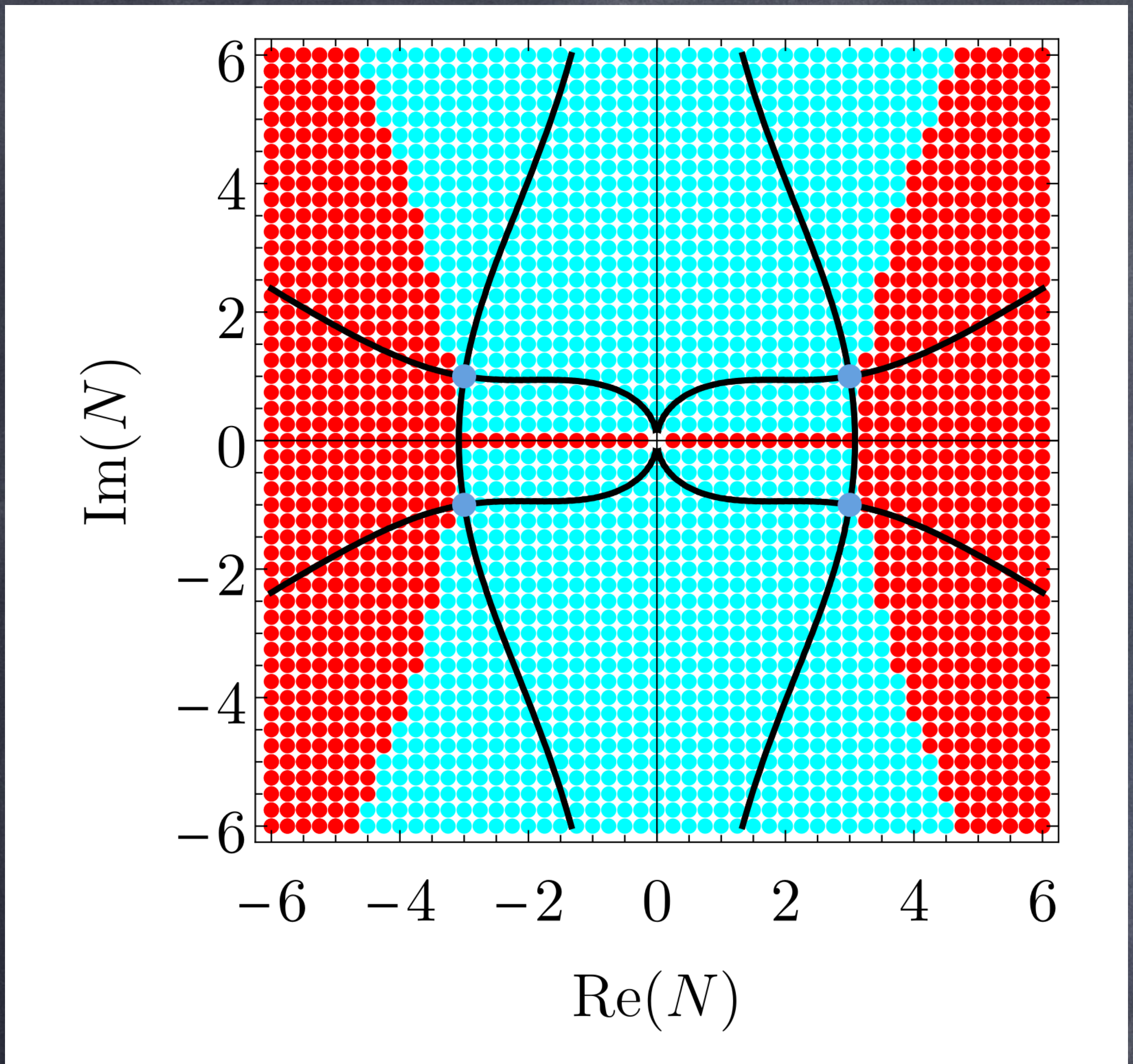

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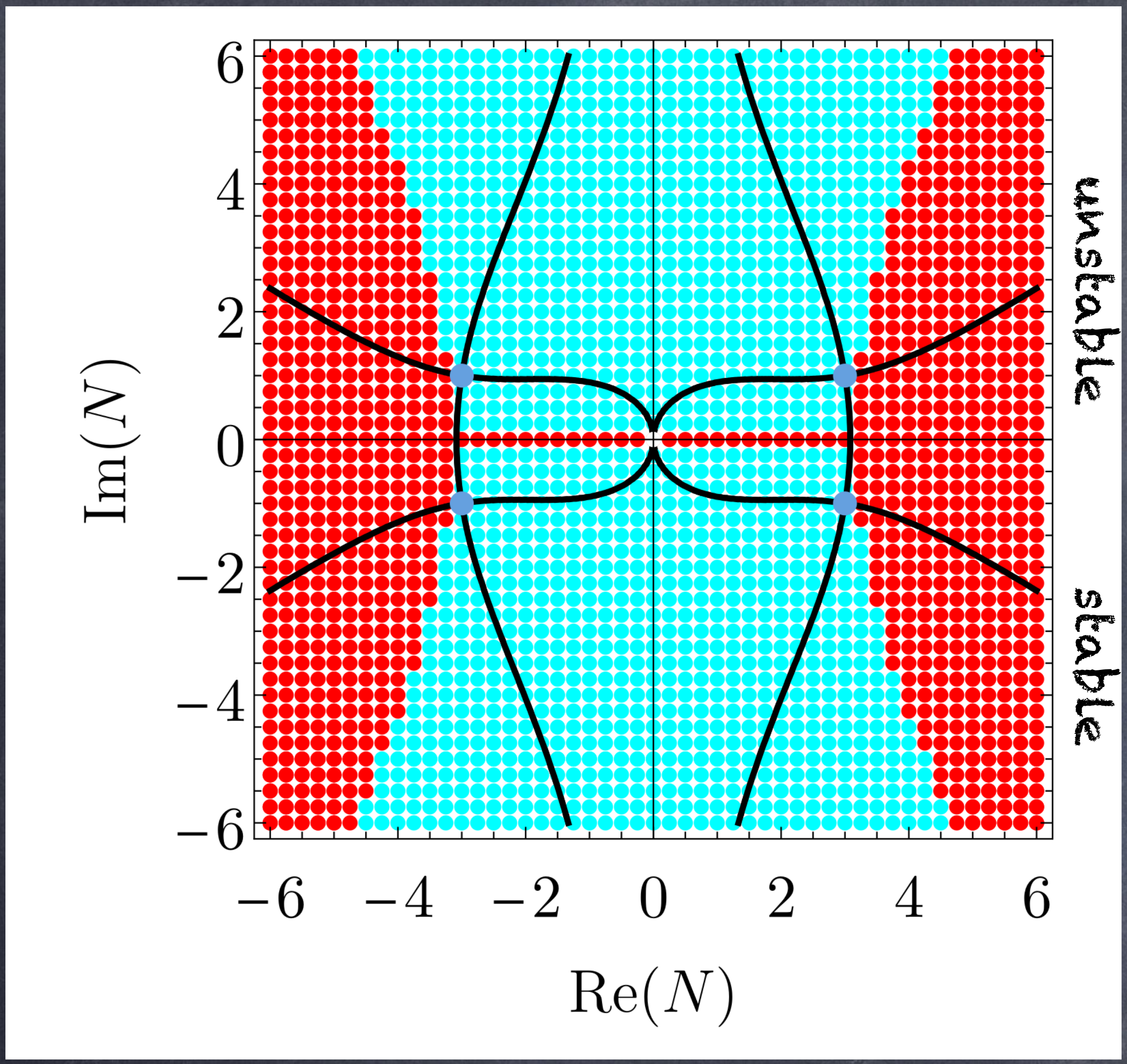

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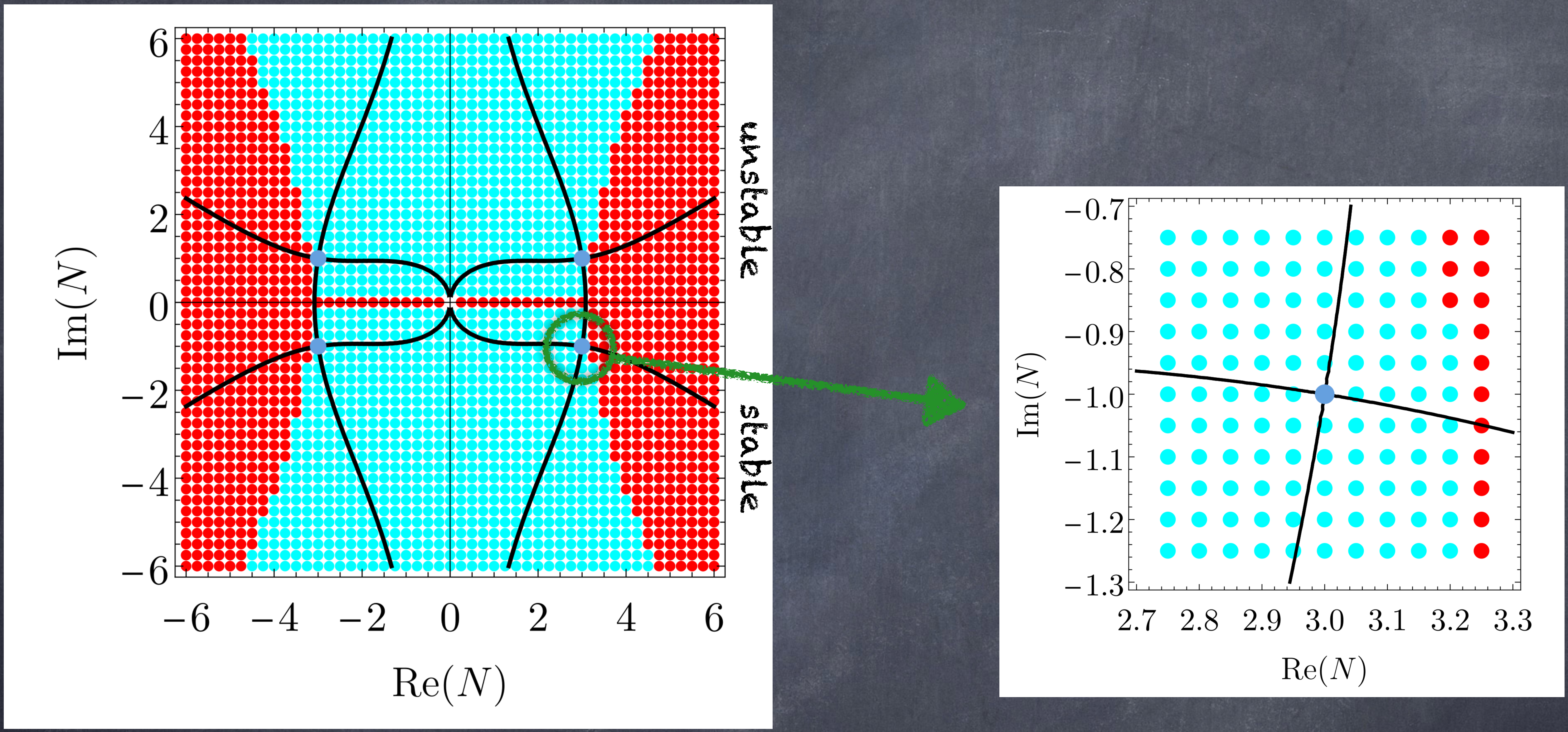

- steepest descent contours cut off; the upper and lower half planes become disconnected
- saddles are not at the boundary this time, but are surrounded by allowable metrics