Stability of Singularity-Resolving Effective Theories of Gravity

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Motivation – Penrose & Hawking ‘singularity’ theorems

E.g., Penrose’s (1965):

- Assume the null convergence condition (a.k.a. NCC; $R_{\mu\nu}k^\mu k^\nu \geq 0 \, \forall \, k^\mu$ null, i.e., $g_{\mu\nu}k^\mu k^\nu = 0$)

$$\mathbf{Ric}(\boldsymbol{k},\boldsymbol{k}) \geq 0 \, \forall \, \boldsymbol{k} \text{ s.t. } \boldsymbol{g}(\boldsymbol{k},\boldsymbol{k}) = 0$$

  - ▶ Assuming general relativity ($G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_\mathrm{N}T_{\mu\nu}$)

$$\boldsymbol{G} := \mathbf{Ric} - \frac{1}{2}\boldsymbol{g}\underbrace{\mathrm{tr}_{\boldsymbol{g}}(\mathbf{Ric})}_{=:R} = 8\pi G_\mathrm{N}\boldsymbol{T} \qquad (c = 1)$$

  the NCC is equivalent to the null energy condition (NEC):

$$T_{\mu\nu}k^\mu k^\nu \geq 0 \quad (\boldsymbol{T}(\boldsymbol{k},\boldsymbol{k}) \geq 0) \quad \forall \, \boldsymbol{k} \text{ null}$$

- Assume $\exists$ a noncompact connected Cauchy surface in $\mathcal{M}$
- Assume $\exists$ a closed trapped surface in $\mathcal{M}$

$\Longrightarrow$ Then $\mathcal{M}$ cannot be null geodesically complete

E.g. for a null geodesic congruence

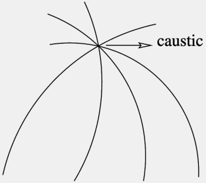
Raychaudhuri's equation for a null hypersurface orthogonal congruence (no vorticity):

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\lambda} = -\frac{1}{2}\Theta^2 - \sigma^2 - \mathbf{Ric}(\boldsymbol{k},\boldsymbol{k}) \overset{\mathrm{NCC}}{\leq} -\frac{1}{2}\Theta^2\,,$$

where $\Theta := \nabla_\mu k^\mu = \boldsymbol{g}(\boldsymbol{\nabla},\boldsymbol{k}) = \mathrm{div}_{\boldsymbol{g}}\boldsymbol{k}$ is the expansion scalar, $\sigma^2 := \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}$ the shear, and λ an affine parameter, from which

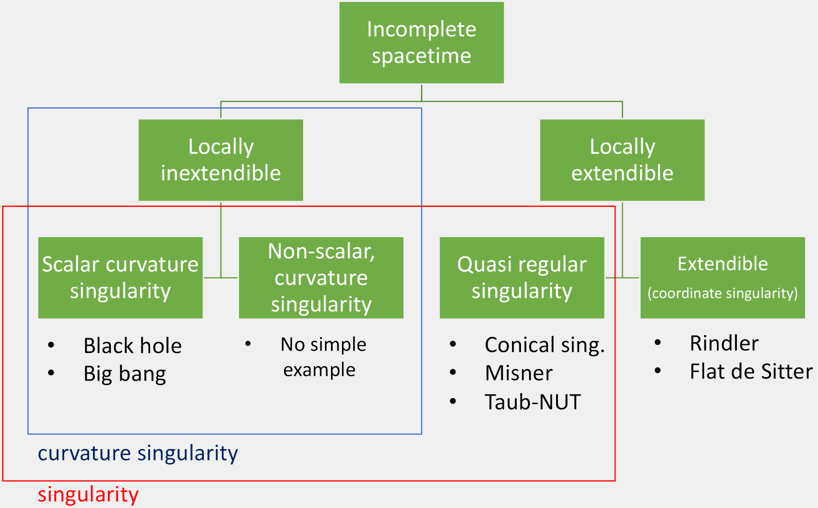
$$\Theta(\lambda)^{-1} \geq \Theta(\lambda=0)^{-1} + \frac{\lambda}{2} \overset{\Theta(\lambda=0)<0}{\Longrightarrow} |\Theta(\lambda)| \to \infty \text{ by } \lambda \leq \frac{2}{|\Theta(\lambda=0)|}\,,$$

so geodesics come together into a caustic in finite affine length
$\longrightarrow$ geodesic incompleteness



From Poisson's relativity textbook

Does not always necessarily imply a singular spacetime



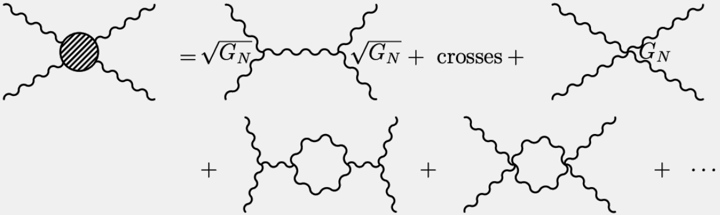
Courtesy of Daisuke Yoshida (Nagoya U.)

Cosmological spacetime extendibility for another talk perhaps e.g., Yoshida & JQ [arXiv:1803.07085], Nomura & Yoshida [2105.05642]

But when there is a scalar curvature singularity...

- At the centre of black holes and at the big bang, $|\mathbf{Riem}^2| = |R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}| \to \infty$, and we are in a strong curvature regime where general relativity's semiclassical physics breaks down $\to$ general relativity is a non-renormalizable quantum field theory
- Consider gravitons propagating and scattering on a Minkowski background: $\boldsymbol{g} = \boldsymbol{\eta} + \sqrt{G_\mathrm{N}}\boldsymbol{h}$, the general relativity action reads

$$S = \int_\mathcal{M} \underbrace{\mathrm{d}^4x\,\sqrt{-\det(\boldsymbol{g})}}_{:=\epsilon=\star 1=\sqrt{-\det(\boldsymbol{g})}\bigwedge_{\nu=0}^{3}\mathrm{d}x^\nu} \frac{R}{16\pi G_\mathrm{N}} \sim \int_\mathcal{M} \mathrm{d}^4x\,\left((\partial h)^2 + \sqrt{G_\mathrm{N}}h(\partial h)^2 + \dots\right)$$



From Hartman's lectures on quantum gravity

Non-renormalizibility continued

- Really an expansion in $\sqrt{G_\mathrm{N}}/\lambda$, where λ is the wavelength of the graviton fluctuations, so the amplitude can be written

$$\mathcal{A}(hh \to hh) \sim \sum_{\mathrm{loops}} \left(\frac{G_\mathrm{N}}{\lambda^2}\right)^{1+\mathrm{number\ of\ loops}}$$

$\Rightarrow$ As $\lambda \lesssim \sqrt{G_\mathrm{N}}$, the perturbative expansion breaks down (strong coupling)

$\Rightarrow$ We cannot trust general relativity as a quantum theory when we reach that regime

$\to$ would require an infinite series of counterterms, schematically

$$S \supset \int_\mathcal{M} \mathrm{d}^4x\, \sqrt{-\det(\boldsymbol{g})} \sum_{n=0}^{\infty} c_{n,p} G_\mathrm{N}^n \sum_{p=0}^{n} \mathrm{scal}\left(\boldsymbol{\nabla}^{2p}\mathbf{Riem}^{2+n-p}\right)$$

$\to$ This motivates going beyond general relativity, especially in strong curvature regimes, such as near singularities

Also, the NCC/NEC might not always hold

- Quantum mechanically, there are well-known (and well-accepted) examples where matter violates the NEC, i.e., $\langle T_{\mu\nu}k^\mu k^\nu\rangle < 0$ (e.g., Casimir effect [Casimir (1948)])

- In quantum gravity, it might only hold in some averaged sense, e.g., the averaged NEC (ANEC) for an achronal null geodesic γ reads

$$\int_\gamma \langle T_{\mu\nu}k^\mu k^\nu\rangle \geq 0$$

  Proved in various contexts, e.g., Wald & Yurtsever (1991), Wall [0910.5751], Kontou & Olum [1507.00297]

- Various other proposals, e.g., the smeared NEC (SNEC):

  see Freivogel and collaborators [1807.03808, 2012.11569, 2111.05772]

$$\underbrace{\int_{-\infty}^{\infty} \mathrm{d}\lambda\, g(\lambda)^2\, \langle T_{\mu\nu}k^\mu k^\nu\rangle|_{x^\mu(\lambda)}}_{\langle\langle T_{\mu\nu}k^\mu k^\nu\rangle\rangle_\tau} \geq -\frac{\mathcal{O}(1)}{G_\mathrm{N}} \underbrace{\int_{-\infty}^{\infty} \mathrm{d}\lambda \left(\frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda}\right)^2}_{1/\tau^2}, \quad \int_{-\infty}^{\infty} \mathrm{d}\lambda\, g(\lambda)^2 = 1$$

The point is...

the assumptions leading to the singularity theorems may well break down

- So are singularities 'avoidable' in quantum gravity?
- Could it even be that spacetime singularities are forbidden?

Various quantum gravity proposals (string theory, loop quantum gravity, etc.) suggest some notion of fundamental, 'minimal' length scale (or maximal curvature scale)

- But studying quantum gravity is very hard! Might require new mathematical tools, e.g., low regularity geometry!

→ What we can try to do instead is to come up with an **effective theory** (of gravity, so modifying general relativity), which could characterize the low-curvature regime of quantum gravity

→ We can then test such a theory (does it yield sensible solutions? is it stable? etc.)

One approach

- They are many approaches to 'modified gravity', motivated by various physical considerations
- One can construct classes of such theories that intrinsically have a maximal curvature scale $\rightarrow$ limiting curvature theories (e.g., Yoshida, JQ, Yamaguchi & Brandenberger [1704.04184], Sakakihara, Yoshida, Takahashi & JQ [2005.10844], but perhaps for another talk!)
- Let me present one such theory: the **Cuscuton** Afshordi, Chung & Geshnizjani [hep-th/0609150]

$$S = \int \mathrm{d}^4x \sqrt{-\det(\boldsymbol{g})} \left(\frac{1}{16\pi G_\mathrm{N}} R \pm M_\mathrm{L}^2 \sqrt{X} - V(\phi) \right) ,$$

$$\text{with } X := -g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = -\boldsymbol{g}(\boldsymbol{\nabla}\phi, \boldsymbol{\nabla}\phi)$$

$$\frac{\delta S}{\delta \boldsymbol{g}} = 0 \Rightarrow \boldsymbol{G} = 8\pi G_\mathrm{N}\boldsymbol{T} = (\rho + p)\boldsymbol{u}\otimes\boldsymbol{u} + p\boldsymbol{g} ,$$

$$\text{where } \boldsymbol{u} = \pm\frac{\boldsymbol{\nabla}\phi}{\sqrt{X}} , \quad \rho = V(\phi) , \quad p = \pm M_\mathrm{L}^2\sqrt{X} - V(\phi)$$

$$\frac{\delta S}{\delta \phi} = 0 \Rightarrow K = M_\mathrm{L}^{-2}\frac{\mathrm{d}V}{\mathrm{d}\phi} , \text{ where } K = \boldsymbol{g}(\boldsymbol{\nabla}, \boldsymbol{u}) = \mathrm{div}_{\boldsymbol{g}}\boldsymbol{u} = \nabla_\mu u^\mu = \pm\nabla_\mu\left(\frac{\nabla^\mu\phi}{\sqrt{X}}\right)$$

i.e., the mean curvature K is the trace of the extrinsic curvature $\boldsymbol{K}$ on a constant-ϕ hypersurface with normal unit vector $\boldsymbol{u}$

$$\delta S = 0 \overset{\delta\phi}{\Rightarrow} K = M_{\mathrm{L}}^{-2}\frac{\mathrm{d}V}{\mathrm{d}\phi}\,,\ \text{where } K = \pm\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\phi}{\sqrt{-\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi}}\right)$$

⇒ Constant-ϕ hypersurfaces are CMC surfaces

⇒ Bounded $\mathrm{d}V/\mathrm{d}\phi$ yields bounded mean curvature $\rightarrow$ may avoid singularities

⇒ ϕ respects a constraint equation, not an evolution equation

e.g., in a flat Friedmann-Lemaître-Robertson-Walker metric background

$$\boldsymbol{g} = -\mathbf{d}t\otimes\mathbf{d}t + a(t)^2\delta_{ij}\mathbf{d}x^i\otimes\mathbf{d}x^j\,,$$

the ϕ equation reduces to

$$\mp\mathrm{sgn}(\dot{\phi})3M_{\mathrm{L}}^2H = \frac{\mathrm{d}V}{\mathrm{d}\phi}\,,$$

where $\dot{} := \mathrm{d}/\mathrm{d}t$ and the Hubble expansion/contraction rate is $H := \dot{a}/a$

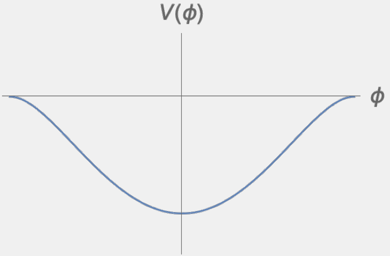
- In comparison, a standard propagating scalar field has equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0$$

More on the Cuscuton

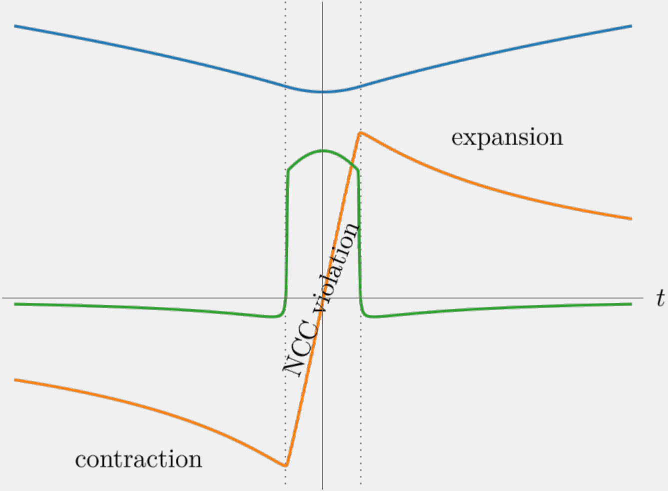
- So on such a cosmological background, we say that there are no new degrees of freedom Afshordi, Chung, Doran & Geshnizjani [astro-ph/0702002], Gomes & Guariento [1703.08226], and more (let me know if you want more references)
- This is an example of a minimal modification of gravity Lin & Mukohyama [1708.03757], ...
- Non-singular spacetimes can be found as solutions, e.g., cosmological bounces Boruah, Kim, Rouben & Geshnizjani [1802.06818], JQ & Yoshida [1911.06040]

At the bounce: $V < 0$, $\frac{\mathrm{d}V}{\mathrm{d}\phi} = 0$, $\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2} > 0$
General relativity limit: $V \to 0$



Example of Cuscuton bounce solution

$$a(t) \qquad H(t) = \frac{\dot{a}}{a} \sim K \qquad \dot{H}(t) \sim -\mathbf{Ric}(\boldsymbol{k}, \boldsymbol{k})$$



Stability

- How can we know that the speculated theory makes any sense?
  - Theoretically, it should be stable and not strongly coupled
  - We should try to find some observational predictions
- Main tool: cosmological perturbation theory ($\boldsymbol{g} = \bar{\boldsymbol{g}} + \delta\boldsymbol{g}$) See Ghazal's previous talk
  E.g., scalar perturbations (in the matter comoving gauge):

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\left(1+2\alpha(t,\vec{x})\right)\mathrm{d}t^2 + 2\partial_i\beta(t,\vec{x})\mathrm{d}t\mathrm{d}x^i + a(t)^2\left(1+2\zeta(t,\vec{x})\right)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

$$\phi(t,\vec{x}) = \bar{\phi}(t) + \delta\phi(t,\vec{x})$$

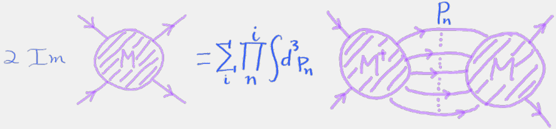
- The lapse and shift perturbations (α, β) can be eliminated by the constraint equations. The Cuscuton perturbation ($\delta\phi$) can also be eliminated since it is governed by a constraint equation. One is left with a single degree of freedom, ζ, known as the **curvature perturbation**
- The action, expanded to 2nd order in the linear perturbations, is reduced to

$$S^{(2)} = \int \mathrm{d}t\mathrm{d}^3\vec{x}\, a z^2\left(\dot{\zeta}^2 - \frac{c_\mathrm{s}^2}{a^2}|\vec{\nabla}\zeta|^2\right) \overset{\delta S^{(2)}/\delta\zeta=0}{\Longrightarrow} \ddot{\zeta} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{z}}{z}\right)\dot{\zeta} - \frac{c_\mathrm{s}^2}{a^2}\nabla^2\zeta = 0$$

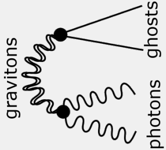
- Stability requires $z^2 > 0$ and $c_\mathrm{s}^2 > 0$

Avoiding quantum 'ghost instability'

- The kinetic term, $az^2\dot{\zeta}^2$, tells you about the propagation; it'd better be > 0
- For scattering amplitudes to respect unitarity (conservation of probability in quantum mechanics), the S matrix has to be unitary ($\mathsf{S}^\dagger\mathsf{S} = 1$, where $\mathsf{S}^\dagger$ is the Hermitian adjoint), from which we can derive the optical theorem (writing $\mathsf{S} = 1 + i\mathsf{M}$, we must have $2\,\mathrm{Im}\,\mathsf{M} = \mathsf{M}^\dagger\mathsf{M}$)
- A 'correct sign' kinetic term respects the theorem, while a 'wrong sign' generally doesn't $\implies$ **unitarity violation**



- At best, negative energy 'ghosts' propagate, leading to a catastrophic **instability of the vacuum** Cline, Jeon & Moore [hep-ph/0311312]



Avoiding classical gradient instability

- Doing a change of variable for ζ, t, and $\vec{x}$, we can write the equation of motion as

$$\frac{\partial^2 \tilde{\zeta}}{\partial \tilde{t}^2} = c_\mathrm{s}^2 \tilde{\nabla}^2 \tilde{\zeta} \quad \longrightarrow \quad \text{wave equation}$$

- We call c_s the sound speed, and we'd better have $c_\mathrm{s}^2 > 0$ for the PDE to be hyperbolic to have a well-posed initial value problem; otherwise PDE is elliptic (or parabolic)

- In Fourier space (let me drop the tildes here)

$$\frac{\mathrm{d}^2 \zeta_{\vec{k}}}{\mathrm{d}t^2} + c_\mathrm{s}^2 |\vec{k}|^2 \zeta_{\vec{k}} = 0 \overset{c_\mathrm{s}^2>0}{\Longrightarrow} \zeta_{\vec{k}}(t) \sim \exp\left(\pm i|\vec{k}| \int \mathrm{d}t\, c_\mathrm{s}\right) \quad \longrightarrow \quad \text{oscillatory}$$

$$c_\mathrm{s}^2 < 0 \implies \zeta_{\vec{k}}(t) \sim \exp\left(\pm |\vec{k}| \int \mathrm{d}t\, |c_\mathrm{s}|\right) \quad \longrightarrow \exists \text{ exponentially growing term}$$

$\implies$ gradient instability

About the Cuscuton again

- It is confirmed that the theory is linearly stable on a cosmological background (including bouncing ones), i.e., it has no ghost and no gradient instability Boruah, Kim & Geshnizjani [1704.01131], Boruah, Kim, Rouben & Geshnizjani [1802.06818], JQ & Yoshida [1911.06040]
- Actually very hard to achieve in general for an effective theory allowing geodesically complete spacetimes Libanov *et al.* [1605.05992], Kobayashi [1606.05831], Cai *et al.* [1610.03400], Creminelli *et al.* [1610.04207], ...
- ► Still, is the theory strongly coupled? Usually hard to avoid in a high-curvature, NCC-violating regime. But the Cuscuton is nice, so probably not (still under investigation) Dehghani, Geshnizjani & JQ
- ► And could there be specific observational signatures of such a Cuscuton bounce? Also under investigation Dehghani, Geshnizjani & JQ

Expand action to 3rd order in perturbations:

$$S^{(3)} = \int \mathrm{d}^3\vec{x}\mathrm{d}t \left(A_1\dot{\zeta}^3 + A_2\zeta\dot{\zeta}^2 + \ldots \right) ; \qquad \text{we want } \frac{A_1\dot{\zeta}^3}{az^2\dot{\zeta}^2} < 1 \,, \ \cdots$$

- $S^{(2)}$ used to compute $\langle\zeta^2\rangle$; $S^{(3)}$ needed to compute $\langle\zeta^3\rangle$
- Equations could also be solved in full (i.e., non-perturbatively) using numerical relativity techniques

Take-home messages

- Classical general relativity singularity theorems are nice, but it is unknown what applies in quantum gravity
- Classical general relativity most likely breaks down before reaching singularities
- If we modify gravity in the high-curvature regime, we can construct theories that avoid singularities altogether
- We can then check if the theories are stable and what predictions they make

Open questions

- About the Cuscuton:
  - Connection to potential ultraviolet completions (i.e., validity up to arbitrarily high energy scales, as in quantum gravity) Afshordi [0907.5201, 1003.4811], Bhattacharyya *et al.* [1612.01824], ...
  - What about black holes? What happens to the singularity there?
  - Can the theory make sense (and be stable) on arbitrary backgrounds?
  - What can the existence of CMC surfaces tell us?
    $\leftarrow$ last couple of questions for mathematicians!
- More generally (to connect mathematicians and physicists):
  - Are there useful tools to study/construct spacetimes with an upper bound on the curvature?
  - Can we construct theories of non-smooth spacetimes? Can there be a smooth continuum limit and what would it be? What tools of non-smooth geometries can we use to start doing physics on such spacetimes?

Thank you for your attention!

Questions?