Extensions of General Relativity with Limited Curvature

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Premise

- General Relativity is a classical theory of gravity, which breaks down at small length scales due to quantum effects — as a quantum field theory, it is non-renormalizable Goroff-Sagnotti [1986]
- Many theories of quantum gravity suggest some notion of minimal, fundamental length scale ℓ_f ($\gtrsim \ell_\mathrm{Pl}$), below which there might be no notion of continuum spacetime — spacetime may be discrete at that scale and the spacetime continuum may emerge above ℓ_f
- E.g., $\ell_\mathrm{s} \sim g_\mathrm{s}^{\frac{2}{2-D}} \ell_\mathrm{Pl}$ in string theory, $A_\mathrm{min} \sim \beta_\text{Barbero-Immirzi} \ell_\mathrm{Pl}^2$ in Loop Quantum Gravity see Hossenfelder [1203.6191] for a review
- It is important to attempt constructing top-down approaches to quantum gravity (further examples: non-commutative geometry, spin foam, **causal dynamical triangulation**, **causal sets**, etc.), but it is a hard problem
- An alternative is to explore a bottom-up, effective theory of quantum gravity, e.g., construct a smooth theory of gravity (modifying general relativity), which has a built-in minimal length scale
- This procedure of building a theory from the bottom-up, imposing expected features of ultraviolet completion, is often applied in particle physics (and more and more in cosmology)

Bounding physical quantities: special relativity as a warm-up

- Start with a **non-relativistic** free particle of mass m and (unbounded) speed v, whose Lagrangian reads

$$L = \frac{1}{2}mv^2$$

- To make it **relativitic**, add a Lagrange multiplier term to limit the speed (special relativity must have $|v| < c = 1$ for a massive particle):

$$L = m\left(\frac{1}{2}v^2 + \chi v^2 - V(\chi)\right)\,, \qquad V(\chi) = \frac{2\chi^2}{1+2\chi}$$

$$\frac{\partial L}{\partial \chi} = 0 \implies v^2 = \frac{\mathrm{d}V}{\mathrm{d}\chi} = 1 - \frac{1}{(1+2\chi)^2} \implies v^2 < 1 \,\forall\, \chi \in (-\infty, \infty)$$

- χ is an auxiliary (non-dynamical) scalar field
- Solving the above for $\chi = \chi(v^2)$ and substituting back in L precisely yields the special relativity Lagrangian:

$$L = m\sqrt{1 - v^2}$$

Same strategy for gravity

- Original idea dates back to the 1980s [Markov, Ginsburg, Mukhanov, Frolov, ...]
- Start with the action for general relativity (the Einstein-Hilbert term proportional to the Ricci scalar R, the trace of the Ricci tensor), but let's add a set of 'curvature-limiting terms' through Lagrange multipliers ($8\pi G_{\mathrm{N}} = 1$):

$$S = \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-\det \boldsymbol{g}} \left(\frac{R}{2} + \sum_{j=1}^{n} \chi_j \mathcal{I}_j(\mathbf{Riem}; \boldsymbol{g}; \boldsymbol{\nabla}) - V(\chi_1, \ldots, \chi_n) \right)$$

$$\frac{\delta S}{\delta \chi_j} = 0 \implies \mathcal{I}_j = \frac{\partial V}{\partial \chi_j}\,; \qquad \left| \frac{\partial V}{\partial \chi_j} \right| < \infty \implies |\mathcal{I}_j| < \infty$$

- The $\mathcal{I}_j(\mathbf{Riem}; \boldsymbol{g}; \boldsymbol{\nabla})$s are scalar curvature-invariant functions, constructed out of the Riemann tensor $\mathbf{Riem}$, contractions with the metric $\boldsymbol{g}$, and covariant derivatives $\boldsymbol{\nabla}$ thereof, which we can bound if the potential V is chosen such that the components of its field-space gradient are everywhere bounded

An example

Mukhanov-Brandenberger [1992], Yoshida-JQ-Yamaguchi-Brandenberger [1704.04184]

- Let's have two Lagrange multiplier terms with ($\mathcal{G}$ is the Gauss-Bonnet curvature invariant, given by $R^2 - 4|\mathbf{Ric}|^2 + |\mathbf{Riem}|^2$):

$$\mathcal{I}_1 = R + \sqrt{R^2 - 6\mathcal{G}}\,, \qquad \mathcal{I}_2 = \sqrt{R^2 - 6\mathcal{G}}$$

- On a flat cosmological background (a.k.a. FRW; $a(t)$ is the scale factor),

$$g = -\mathrm{d}t^2 + a(t)^2\mathrm{d}\mathbf{x}^2\,,$$

  these reduce to

$$\mathcal{I}_1 \propto \left(\frac{\mathrm{d}\ln a}{\mathrm{d}t}\right)^2 = H^2\,, \qquad \mathcal{I}_2 \propto \frac{\mathrm{d}^2\ln a}{\mathrm{d}t^2} = \frac{\mathrm{d}H}{\mathrm{d}t}\,,$$

  so one can construct solutions for which the metric is (C^2) non-singular

- Mukhanov-Brandenberger showed examples where the solutions where smoothly non-singular, approaching de Sitter or Minkowski spacetimes asymptotically
- Moreover, background and linear perturbations about the background have at most second-order equations of motion ✓

The challenge

Yoshida-JQ-Yamaguchi-Brandenberger [1704.04184]

- The previous theory is equivalent to a particular form of $f(R,\mathcal{G})$ gravity, where
$$S=\int_{\mathcal{M}}\mathrm{d}^4x\,\sqrt{-\det\boldsymbol{g}}\,f(R,\mathcal{G})$$
- The problem is that this theory is well-known to be problematic: some corner of phase space will always remain unstable (ghost or gradient instability)
see my previous seminar at Fields for an introduction to this topic
- Moreover, as soon as one introduces the slightest anisotropy, a ghost (otherwise absent about FRW) always appears De Felice-Tanaka [1006.4399]
- Introducing higher spacetime curvature terms in the gravitational effective action is ubiquitous (especially toward high-curvature scales) and such terms can be constrained e.g., Caron-Huot+ [2201.06602], de Rham+ [2203.06805], but often, either the theory remains singular, either it is unstable e.g., Yoshida-Brandenberger [1801.05070]

A modified approach: the setup

- Let us perform a $3+1$ splitting of the spacetime with timelike unit vector $\boldsymbol{n}$ [$\boldsymbol{g}(\boldsymbol{n},\boldsymbol{n}) = \boldsymbol{n}\cdot\boldsymbol{n} = -1$], normal to a hypersurface Σ with induced spatial metric $\boldsymbol{\gamma}$ [$\boldsymbol{\gamma} = \boldsymbol{g} + \boldsymbol{n}\otimes\boldsymbol{n}$]
- The metric can be written in terms of the lapse N and shift $\boldsymbol{\beta}$:

$$g = -N\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + \beta^i\mathrm{d}t)(\mathrm{d}x^j + \beta^j\mathrm{d}t)$$

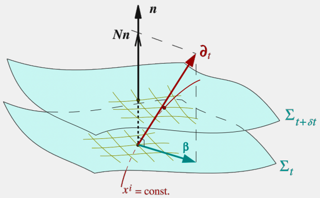

Figure from Gourgoulhon

- The extrinsic curvature (second fundamental form) on Σ and its trace are

$$\boldsymbol{K} = \boldsymbol{\nabla}\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{\nabla}_{\boldsymbol{n}}\boldsymbol{n}\,, \qquad K = \boldsymbol{\nabla}\cdot\boldsymbol{n}$$

- Spacetime curvature (${}^{(4)}\mathbf{Riem}$, ${}^{(4)}\mathbf{Ric}$, ${}^{(4)}R$, ...) can be fully rewritten in terms of those quantities ($\boldsymbol{K}$, ${}^{(3)}R$, ...)

A modified approach: the idea and construction

Limiting Extrinsic Curvature Theory, Sakakihara-Yoshida-Takahashi-JQ [2005.10844]

- In the $3+1$ splitting of the geometry, let us assume time remains smooth, but let us try to bound spatial curvature, i.e., 3-dimensional spatial curvature invariants rather 4-d spacetime quantities:

$$S = S_{\mathrm{gen.\,rel.}} + \int \mathrm{d}t \int_\Sigma \mathrm{d}^3x\, N \sqrt{\det \boldsymbol{\gamma}} \left(\sum_{j=1}^{n} \chi_j \mathcal{I}_j(\boldsymbol{K}; \boldsymbol{\gamma}; \mathbf{D}) - V(\chi_1, \ldots, \chi_n) \right)$$

- We need to introduce a field (e.g., a scalar ϕ or a vector $\boldsymbol{A}$) that defines our spatial hypersurface (very common when Lorentz invariance is [spontaneously] broken, e.g., in cosmology):

$$\boldsymbol{n} = \begin{cases} \boldsymbol{\nabla}\phi \\ \boldsymbol{A} \end{cases}$$

- Let us take a vector field from here on. Then, we must enforce the normalization $\boldsymbol{A}\cdot\boldsymbol{A} = -1$ through an additional constraint, thus introducing an additional Lagrange multiplier λ term in the action:

$$\int \mathrm{d}^4x\, \sqrt{-\det \boldsymbol{g}}\, \lambda(\boldsymbol{A}\cdot\boldsymbol{A} + 1)$$

A simple realization

Sakakihara-Yoshida-Takahashi-JQ [2005.10844]

- Let us consider a single curvature invariant, $K = \boldsymbol{\nabla} \cdot \boldsymbol{n} = \boldsymbol{\nabla} \cdot \boldsymbol{A}$:

$$S = \int \mathrm{d}t \int_\Sigma \mathrm{d}^3x\, N\sqrt{\det \boldsymbol{\gamma}} \Big(\underbrace{\frac{1}{2}\left(|\boldsymbol{K}|^2 - K^2 + {}^{(3)}R\right)}_{\text{gen. rel.}} + \lambda(\boldsymbol{A}\cdot\boldsymbol{A}+1) + \chi K - V(\chi) \Big)$$

- A potential with $|\mathrm{d}V/\mathrm{d}\chi| < \infty$ ensures $|K| < \infty$ thanks to

$$\frac{\delta S}{\delta \chi} = 0 \implies K = \frac{\mathrm{d}V}{\mathrm{d}\chi}$$

- We recognize this constraint as the **Cuscuton** constraint, and the theory is indeed equivalent to the Cuscuton Afshordi-Chung-Geshnizjani [hep-th/0609150]; see my previous seminar at Fields

$$\frac{\delta S}{\delta \boldsymbol{A}} = 0 \implies \boldsymbol{A} = \frac{1}{2\lambda}\boldsymbol{\nabla}\chi$$

$$\implies S = S_{\text{gen. rel.}} + \int \mathrm{d}^4x\, \sqrt{-\det \boldsymbol{g}}\left(\pm\sqrt{-\boldsymbol{\nabla}\chi\cdot\boldsymbol{\nabla}\chi} - V(\chi)\right)$$

- On a cosmological background, $K = 3H$, so a non-singular (bouncing) universe can follow, with stable linear inhomogeneities Boruah-Kim-Rouben-Geshnizjani [1802.06818], JQ-Yoshida [1911.06040]

Relation to other theories

Sakakihara-Yoshida-Takahashi-JQ [2005.10844]

- Solving the constraint $K = \mathrm{d}V/\mathrm{d}\chi$ for $\chi = \chi(K)$ (which one can do if $\mathrm{d}^2V/\mathrm{d}\chi^2 \neq 0$), then with the Legendre transform

$$F(K) = K\chi(K) - V(\chi(K))\,,$$

  we can rewrite the action as

$$S = \int \mathrm{d}t \int_\Sigma \mathrm{d}^3x\, N\sqrt{\det\boldsymbol{\gamma}}\left(\frac{1}{2}\left(|\boldsymbol{K}|^2 - K^2 + {}^{(3)}\!R\right) + F(K)\right)$$

- This makes it manifest that the theory modifies general relativity (changes its K^2 term), yet does not introduce any additional degrees of freedom (only 2 tensor pert.) — it is a subclass of such theories that only minimally modify gravity e.g., Lin-Mukohyama [1708.03757], Mukohyama-Noui [1905.02000], Gao-Yao [1910.13995]
- Other theories are of the above $F(K)$ form, including the low-energy limit of **Hořava-Lifshitz** gravity, a proposal for quantum gravity explicitly breaking Lorentz invariance — the equivalence with the Cuscuton is manifest for a particular potential Afshordi [0907.5201], Bhattacharyya+ [1612.01824], Chagoya-Tasinato [1805.12010]
- Other choices of $F(K)$ result in the same equations as those of **Loop Quantum Cosmology** or **Group Field Theory**, known to predict a non-singular (bouncing) universe Chamseddine-Mukhanov [1612.05860], Bodendorfer-Schäfer-Schliemann [1703.10670], Langlois+ [1703.10812], de Cesare [1812.06171,1904.02622]

Conclusions and discussion

- One might expect some notion of limited curvature in quantum gravity
  - Can we find a consistent low-energy effective theory, which manifests this property?
- Limiting Extrinsic Curvature Theory is a nice setup to build such theories, especially in the context where a preferred frame is selected (e.g., cosmology)
- A simple realization leads to the Cuscuton, which connects with different proposals for quantum gravity — furthermore, the theory passes several sanity checks and has interesting phenomenology (e.g., bouncing cosmology)
- Extensions and further pheno: one can bound more than one curvature invariant, e.g., adding a term bounding $|\boldsymbol{K}|^2 - K^2/3$ enables one to limit the growth of anisotropies in Bianchi spacetimes Sakakihara-Yoshida-Takahashi-JQ [2005.10844] $\Longrightarrow$ evades the BKL approach to singularities
- Future directions: observational prediction in the early universe? non-perturbative stability (using numerical relativity)?

Thank you for your attention!

Questions?