Cuscuton Gravity as a Classically Stable Limiting Curvature Theory

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Motivation

- GR + normal matter $\implies$ inevitable singularities Penrose (1965), Hawking (1967), ...
- Even inflationary cosmology (within GR) is inevitably past incomplete and often inextendible Borde & Vilenkin (1994), Border et al. (2003), Yoshida & JQ (2018), ...
- One would thus like to build a theory that is free of these singularities $\implies$ one has to go beyond classical GR
- Singularity resolution $\iff$ modify GR (modified gravity, quantum gravity) or matter (energy conditions)

Not an easy task...

- A popular avenue: consider a generic scalar-tensor theory, e.g., Horndeski, with many free functions
- Those admit non-singular cosmological background solutions
- However, perturbations are often plagued with instabilities: **ghosts and gradient instabilities** $\longrightarrow$ indications for a no-go theorem Libanov et al. (2016), Kobayashi (2016), Creminelli et al. (2016), Cai et al. (2017), ...
- Very few ways of evading the no-go theorem and often at some costs Ijjas & Steinhardt (2016,2017), Cai & Piao (2017), Cai et al. (2017), Kolevator et al. (2017), Dobre et al. (2017), Mironov et al. (2018,2019), Ye & Piao (2019), Banerjee et al. (2019), ...

Limiting curvature

- Different approach to singularity resolution: impose constraint equations that ensure the boundedness of curvature
  $\Longrightarrow$ **limiting curvature**
- Example of implementation Mukhanov & Brandenberger (1992), Brandenberger et al. (1993)

$$S = S_{\mathrm{EH}} + \int \mathrm{d}^4x \sqrt{-g} \left[\sum_{i=1}^{n} \varphi_i I_i(\mathbf{Riem}, \boldsymbol{g}, \boldsymbol{\nabla}) - V(\varphi_1, ..., \varphi_n) \right]$$

$$\delta_{\varphi_i} S = 0 \implies I_i = V_{,\varphi_i}$$

$$|V_{,\varphi_i}| < \infty \; \forall \varphi_i \implies \text{bounded curvature}$$

- Concrete model (e.g., $n = 2$)

$$I_1 = \sqrt{12 R_{\mu\nu} R^{\mu\nu} - 3R^2} \overset{\mathrm{FRW}}{\propto} \dot{H} \,, \qquad I_2 = R + I_1 \overset{\mathrm{FRW}}{\propto} H^2$$

- $\longrightarrow$ non-singular background cosmology, but severe instabilities Yoshida, JQ et al. (2017)

- Another implementation of limiting curvature: mimetic gravity Chamseddine & Mukhanov (2013,2017), ...

$$S = S_{\mathrm{EH}} + \int \mathrm{d}^4x\,\sqrt{-g}\,[\lambda(\partial_\mu\phi\partial^\mu\phi + 1) + \chi\Box\phi - V(\chi)]$$

$$\delta_\lambda S = 0 \implies \partial_\mu\phi\partial^\mu\phi = -1$$

$$\delta_\chi S = 0 \implies \Box\phi = V_{,\chi}$$

- E.g., $\phi = t \implies \Box\phi = 3H$, so bounding $V_{,\chi}$ ensures H does not blow up

- Yet, mimetic gravity suffers from (gradient) instabilities Ijjas et al. (2016), Firouzjahi et al. (2017), Langlois et al. (2019), ...

Cuscuton gravity

- Setup: GR + non-dynamical scalar field ϕ on cosmological background
- Subclass of 'minimally-modified gravity' (modified gravity with only 2 d.o.f., i.e., the 2 tensor modes of GR) Lin & Mukohyama (2017), Carballo-Rubio et al. (2018), Aoki et al. (2018,2019), Lin (2019), Mukohyama & Noui (2019)
- Original implementation: start with k-essence theory Afshordi et al. (2007)

$$S = S_{\rm EH} + \int \mathrm{d}^4x\,\sqrt{-g}P(X,\phi)\,, \qquad X \equiv -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$$

$$\delta_\phi S = 0 \stackrel{\rm FRW}{\Longrightarrow} (P_{,X} + 2XP_{,XX})\ddot{\phi} + 3HP_{,X}\dot{\phi} + P_{,X\phi}\dot{\phi}^2 - P_{,\phi} = 0$$

- Requiring $P_{,X} + 2XP_{,XX} = 0$ sets

$$P(X,\phi) = c_1(\phi)\sqrt{|X|} + c_2(\phi)$$

- Rescaling ϕ, we can write

$$\mathcal{L}_{\rm cuscuton} = \pm M_L^2\sqrt{2X} - V(\phi)\,, \qquad \partial_\mu\phi\ \mathrm{timelike}$$

- EOM becomes a constraint equation:

$$\mp \mathrm{sgn}(\dot{\phi}) 3 M_L^2 H = V_{,\phi}$$

- $\longrightarrow$ limiting curvature

$$M_L^2 K = V_{,\phi}\,, \qquad K = \nabla_\mu u^\mu\,, \qquad u_\mu = \pm\frac{\partial_\mu \phi}{\sqrt{2X}}$$

- Incompressible perfect fluid

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}\,, \quad \rho = 2XP_{,X} - P = V\,, \quad p = P$$

$$c_s^2 = \frac{p_{,X}}{\rho_{,X}} = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}} \to \infty$$

- Other interesting properties:
    - forms no caustics de Rham & Motohashi (2017)
    - geometrical interpretation Chagoya & Tasinato (2017)
    - new symmetries Pajer & Stefanyszyn (2019), Grall et al. (2019)
    - and more

- Fluctuations do not propagate:

$$\delta g_{ij} = -2a^2\zeta\delta_{ij} \implies S^{(2)}_{\mathrm{scalar}} = \int \mathrm{d}t\mathrm{d}^3\mathbf{x}\, a^3 \left(\mathcal{G}_S\dot{\zeta}^2 - \frac{\mathcal{F}_S}{a^2}(\vec{\nabla}\zeta)^2\right)\,,$$

$$\text{where} \qquad \mathcal{G}_S = \frac{X}{H^2}(P_{,X} + 2XP_{,XX}) = 0\,, \qquad \mathcal{F}_S = -M_{\mathrm{pl}}^2\dot{H}/H^2\,;$$

$$S^{(2)}_{\mathrm{scalar}} = \int \mathrm{d}t\mathrm{d}^3\mathbf{x}\, a^3\mathcal{G}_S \left(\dot{\zeta}^2 - \frac{c_S^2}{a^2}(\vec{\nabla}\zeta)^2\right)\,, \qquad c_S^2 = \frac{\mathcal{F}_S}{\mathcal{G}_S} \to \infty$$

- $\longrightarrow$ no-go theorem in Horndeski theory does not apply
- But what happens if $H = 0$, e.g., through a bounce?

Cusctuon gravity with matter

- Consider the addition of a massless scalar field

$$\mathcal{L} = \mathcal{L}_{\mathrm{EH}} \pm M_L^2\sqrt{2X} - V(\phi) - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi$$

$$\overset{\mathrm{FRW}}{\Longrightarrow} \ 3M_{\mathrm{pl}}^2H^2 = \frac{1}{2}\dot{\chi}^2 + V(\phi)\,, \quad 2M_{\mathrm{pl}}^2\dot{H} = -\dot{\chi}^2 \mp M_L^2|\dot{\phi}|$$

- Choose '$-$' sign in $\mathcal{L}_{\mathrm{cuscuton}}$
- NEC violation:

$$M_L^2|\dot{\phi}| > \dot{\chi}^2 \implies 2M_{\mathrm{pl}}^2\dot{H} = -\dot{\chi}^2 + M_L^2|\dot{\phi}| > 0$$

- Requirement for a bounce:

$$\mathrm{sgn}(\dot{\phi})3M_L^2H = V_{,\phi} \implies 3M_L^2\dot{H} = V_{,\phi\phi}|\dot{\phi}|$$
$$V_{,\phi\phi} > 0 \implies \dot{H} > 0$$

Cosmological perturbations

- Consider the comoving gauge w.r.t. ϕ, so $\delta\phi = 0$, but $\chi(t, \mathbf{x}) = \chi(t) + \delta\chi(t, \mathbf{x})$ and

$$\mathrm{d}s^2 = -(1 + 2\Phi)\mathrm{d}t^2 + 2a\partial_i B \mathrm{d}x^i \mathrm{d}t + a^2(1 - 2\Psi)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j$$

- Perturbed Hamiltonian and momentum constraints in Fourier space (setting $M_{\mathrm{pl}} = 1$):

$$(\dot{\chi}^2/2 - 3H^2)\Phi_k + H(k/a)^2 B_k + 3H\dot{\Psi}_k + (k/a)^2\Psi_k - \dot{\chi}\delta\dot{\chi}_k = 0$$
$$2H\Phi_k - 2\dot{\Psi}_k - \dot{\chi}\delta\chi_k = 0$$

- $\longrightarrow$ need to divide by H (in particular when $H = 0$) to eliminate Φ_k and B_k
  $\longrightarrow$ potential divergences

- After simplification,

$$S^{(2)}_{\mathrm{scalar}} = \int \mathrm{d}t\mathrm{d}^3\mathbf{k}\, a z^2 \left(\dot{\zeta}_k^2 - c_\mathrm{s}^2 \frac{k^2}{a^2}\zeta_k^2\right)\,, \qquad \zeta_k = -\Psi_k - \frac{H}{\dot{\chi}}\delta\chi_k\,,$$

where

$$z^2 = a^2 \frac{\dot{\chi}^2(k^2/a^2 + 3\dot{\chi}^2/2)}{(k/a)^2H^2 + \dot{\chi}^2(3H^2 + \underbrace{\dot{H} + \dot{\chi}^2/2}_{=M_L^2|\dot{\phi}|/2})/2} > 0\,, \quad \checkmark$$

$$c_\mathrm{s}^2 = \frac{H^4k^4/a^4 + A_2k^2/a^2 + A_0}{H^4k^4/a^4 + B_2k^2/a^2 + B_0} \overset{k\to\infty}{\longrightarrow} 1 > 0\,, \quad \checkmark$$

with

$$A_2 \equiv \dot{\chi}^2/2\left(12H^2 + 3\dot{H} + \dot{\chi}^2/2\right) + 2\dot{H}^2 - H\ddot{H}$$

$$A_0 \equiv \left(\dot{\chi}^2/2\right)^2\left(15H^2 + \dot{H} - \dot{\chi}^2/2\right) - \dot{\chi}^2/2\left(12H^2\dot{H} - 2\dot{H}^2 + 3H\ddot{H}\right)$$

$$B_2 \equiv \dot{\chi}^2/2\left(6H^2 + \dot{H} + \dot{\chi}^2/2\right)\,,\; B_0 \equiv 3\left(\dot{\chi}^2/2\right)^2\left(3H^2 + \dot{H} + \dot{\chi}^2/2\right)$$

- Note, however,

$$z^2 \overset{k\to\infty}{\longrightarrow} a^2\dot{\chi}^2/H^2 \overset{H\to 0}{\longrightarrow} \infty$$

Switch gauge

- Spatially flat ($\Psi^S = 0$):

$$\Phi_k^S = -\frac{d}{dt}(\zeta_k/H) + \mathcal{O}(H^0)\,,\ aB_k^S = \zeta_k/H + \mathcal{O}(H^0)\,,$$

$$\delta\chi_k^S = -\dot{\chi}\zeta_k/H + \mathcal{O}(H^0)\,,\ \delta\phi_k^S = -\dot{\phi}\zeta_k/H + \mathcal{O}(H^0)$$

$$\implies \text{ill defined at } H = 0$$

- Back to comoving gauge w.r.t. ϕ ($\delta\phi^\phi = 0$):

$$\Phi_k^\phi = \Phi_k^S - \frac{d}{dt}(\delta\phi_k^S/\dot{\phi}) = -\frac{4}{1 + 3\dot{\chi}^2 a^2/2k^2}\zeta_k + \mathcal{O}(H)$$

$$aB_k^\phi = aB_k^S + \delta\phi_k^S/\dot{\phi} = -\frac{3a^2\dot{\chi}^2}{M_L^2 k^2 \dot{\phi}}\dot{\zeta}_k + \mathcal{O}(H)$$

$$\Psi_k^\phi = H\delta\phi_k^S/\dot{\phi} = -\zeta_k + \mathcal{O}(H)$$

$$\delta\chi_k^\phi = \delta\chi_k^S - \dot{\chi}\delta\phi_k^S/\dot{\phi} = -\frac{2\dot{\chi}}{M_L^2\dot{\phi}}\dot{\zeta}_k + \mathcal{O}(H)$$

- $\Longrightarrow$ divergences exactly cancel out to yield well-defined perturbations at $H = 0$

  $\Longrightarrow$ valid perturbed action $\mathcal{L}_{\mathrm{s}}^{(2)} = az^2(\dot{\zeta}_k^2 - c_{\mathrm{s}}^2 k^2 \zeta_k^2/a^2)$

- Comoving gauge w.r.t. χ ($\delta\chi^\chi = 0$):

$$\Phi_k^\chi = \Phi_k^S - \frac{d}{dt}\left(\frac{\delta\chi_k^S}{\dot{\chi}}\right) = -\cancel{\frac{d}{dt}\left(\frac{\zeta_k}{H}\right)} - \cancel{\frac{d}{dt}\left(-\frac{\zeta_k}{H}\right)} + \mathcal{O}(H^0)$$

$$aB_k^\chi = aB_k^S + \frac{\delta\chi_k^S}{\dot{\chi}} = \cancel{\frac{\zeta_k}{H}} + \left(-\cancel{\frac{\zeta_k}{H}}\right) + \mathcal{O}(H^0)$$

$$\Psi_k^\chi = H\frac{\delta\chi_k^S}{\dot{\chi}} = \cancel{H}\left(-\frac{\zeta_k}{\cancel{H}}\right) + \mathcal{O}(H^0)$$

$$\delta\phi_k^\chi = \delta\phi_k^S - \dot{\phi}\frac{\delta\chi_k^S}{\dot{\chi}} = \cancel{-\frac{\dot{\phi}}{H}\zeta_k} \cancel{-\dot{\phi}\left(-\frac{\zeta_k}{H}\right)} + \mathcal{O}(H^0)$$

$\longrightarrow$ all finite at $H = 0$

- Newtonian gauge ($B^N = 0$):

$$\Phi_k^N = \Phi_k^S + \frac{d}{dt}(aB_k^S) = -\frac{d}{dt}\left(\frac{\zeta_k}{H}\right) + \frac{d}{dt}\left(\frac{\zeta_k}{H}\right) + \mathcal{O}(H^0)$$

$$\Psi_k^N = -aHB_k^S = -H\frac{\zeta_k}{H} + \mathcal{O}(H^0)$$

$$\delta\phi_k^N = \delta\phi_k^S + a\dot{\phi}B_k^S = -\dot{\phi}\frac{\zeta_k}{H} + \dot{\phi}\frac{\zeta_k}{H} + \mathcal{O}(H^0)$$

$$\delta\chi_k^N = \delta\chi_k^S + a\dot{\chi}B_k^S = -\dot{\chi}\frac{\zeta_k}{H} + \dot{\chi}\frac{\zeta_k}{H} + \mathcal{O}(H^0)$$

$$\longrightarrow \text{all finite at } H = 0$$

So what really goes on close to $H = 0$?

- Take the limit $H \to 0$ first and then $k \to \infty$:

$$S_{\rm s}^{(2)} \overset{H\approx 0}{\simeq} \frac{4}{M_L^2} \int \mathrm{d}t\mathrm{d}^3\mathbf{k}\, \frac{ak^2}{|\dot{\phi}|} \left[\dot{\zeta}_k^2 - \left(1 + \frac{\dot{H}}{\dot{\chi}^2}\right) \frac{k^2}{a^2}\zeta_k^2\right], \qquad \text{(UV)}$$

- $\longrightarrow$ confirms that there is no divergence
- Sound speed when $H \approx 0$ (reinserting $M_{\rm pl}$, defining $m^2 \equiv V_{,\phi\phi}|_{\rm bounce}$):

$$c_{\rm s}^2 \overset{\frac{k}{a} \ll \mathcal{O}(\dot{\chi})}{\sim} -\frac{1}{3} + \frac{4m^2 M_{\rm pl}^2}{3(3M_L^4 - 2m^2 M_{\rm pl}^2)} \in (0, 1] \quad \text{if} \quad \frac{1}{2} < \frac{m^2 M_{\rm pl}^2}{M_L^4} \leq 1$$

$$c_{\rm s}^2 \overset{\frac{k}{a} \gtrsim \mathcal{O}(\dot{\chi})}{\sim} 1 + \frac{4m^2 M_{\rm pl}^2}{3M_L^4 - 2m^2 M_{\rm pl}^2} \sim \mathcal{O}(1 - 10)$$

- $\longrightarrow$ superluminality near $H \approx 0$ for mid- to large-k modes

Evolution of ζ_k in the IR in a bounce phase

- The evolution of ζ_k in the IR through a bounce phase links perturbations from a contracting phase (scale invariant?) to the CMB
- For $k \to 0$,

$$\ddot{\zeta} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{z}}{z}\right)\dot{\zeta} = 0 \implies \zeta = \mathrm{const.} \ \text{ and } \ \zeta(t) \propto \int^t \frac{\mathrm{d}t}{az^2}$$

- Can ζ undergo significant amplification? Generally not the case, but if so, possibly important non-Gaussianities generated Battarra et al. (2014), JQ et al. (2015)
- In general, if $z \propto a$ (constant EoS), then $\Delta\zeta < \dot{\zeta}_i (a_i/a_B)^3 \Delta t$
- Here,

$$z^2 \overset{k\to 0}{\simeq} \frac{3a^2\dot{\chi}^2/M_{\mathrm{pl}}^2}{3H^2 + \dot{H} + \dot{\chi}^2/2M_{\mathrm{pl}}^2} \not\propto a^2$$

- One finds $\Delta\zeta < \dot{\zeta}_i (a_i/a_B)^3 \mathcal{E} \Delta t$ with

$$\mathcal{E} = \frac{1 + 3\left(1 - \frac{3}{2}\frac{M_L^4}{m^2 M_{\rm pl}^2}\right)\left(\frac{a_i}{a_B}\right)^3 \left(\frac{H_i^2}{\dot{H}_B} + \frac{1}{3}\right)}{1 + 3\left(1 - \frac{3}{2}\frac{M_L^4}{m^2 M_{\rm pl}^2}\right)\left(\frac{a_i}{a_B}\right)^6 \frac{H_i^2}{\dot{H}_B}}$$

- Recall

$$1 - \frac{3}{2}\frac{M_L^4}{m^2 M_{\rm pl}^2} \sim \mathcal{O}(1)\,, \qquad \text{so } \mathcal{E} \gg 1 \text{ is impossible}$$

- $\longrightarrow$ large wavelength curvature perturbations passing through a bounce cannot receive more amplification than $\mathcal{O}(\dot{\zeta}_i (a_i/a_B)^3 \Delta t)$

Take-home messages

- Cuscuton gravity is a limiting curvature theory (bounds the extrinsic curvature)
- One can resolve cosmological singularities
- Cosmological perturbations are stable: no ghost and no gradient instability
- Sound speed becomes superluminal, only in the UV and near the bounce
- Curvature perturbations remain constant in the IR through a bounce
- Spatially-flat gauge ill defined at $H = 0$
- Divergences at $H = 0$ cancel out in other gauges
- Conclusions transpose to extended cuscuton model (see additional slides)

Future directions

- Strong coupling problem? de Rham & Melville (2017)
  Non-Gaussianities? JQ et al. (2015)

- Quantization and UV completion?

- New generalized limiting curvature? Instead of

$$\mathcal{L}_{\mathrm{lim}} = \sum_{i=1}^{n} \varphi_i I_i(\mathbf{Riem}, \boldsymbol{g}, \boldsymbol{\nabla}) - V(\varphi_1, ..., \varphi_n)$$

  consider

$$\mathcal{L}_{\mathrm{lim}} = \sum_{i=1}^{n} \varphi_i I_i(\boldsymbol{K}, \boldsymbol{h}, \boldsymbol{D}) - V(\varphi_1, ..., \varphi_n)$$

- Cuscuton $\equiv$ vector mimetic?

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Additional slides

Example of bouncing solution

- Let $\phi = 0$ correspond to the bounce point. Then consider

$$V(\phi) \simeq V_0 + \frac{1}{2} m^2 \phi^2 \,, \qquad m^2 = V_{,\phi\phi}(\phi = 0) > 0$$

$$\overset{\text{EOM}}{\Longrightarrow} \; \phi \simeq \frac{3M_L^2}{m^2} H \,, \qquad 3\tilde{M}^2 H^2 \simeq \frac{1}{2}\dot{\chi}^2 + V_0 \,, \qquad 2\tilde{M}^2 \dot{H} \simeq -\dot{\chi}^2$$

$$V_0 < 0 \,, \qquad \tilde{M}^2 \equiv M_{\text{pl}}^2 \left(1 - \frac{3}{2} \frac{M_L^4}{m^2 M_{\text{pl}}^2} \right) < 0 \implies \frac{m^2 M_{\text{pl}}^2}{M_L^4} < \frac{3}{2}$$

- Taylor series solution:

$$a(t) \simeq a_0 \left(1 + \frac{V_0}{2\tilde{M}^2} t^2 \right) \,, \quad H(t) \simeq \frac{V_0}{\tilde{M}^2} t \,, \quad \dot{H} \simeq \frac{V_0}{\tilde{M}^2}$$

- For full solution, see Boruah et al. (2018)

Extended cuscuton

- Rather than starting with $P(X, \phi)$, start with Horndeski or even beyond-Horndeski theory, and impose Iyonaga et al. (2018)
  1. the background EOM to be at most a first-order constraint equation
  2. and the kinetic term of scalar perturbations to vanish

  $\longrightarrow$ extended cuscuton $\supset$ original cuscuton

- Alternatively, in the ADM formalism, one can construct a Hamiltonian, satisfying the appropriate conditions for the theory to propagate at most 2 gravitational d.o.f. and remaining invariant under 3-D diffeomorphisms (but possibly breaking time diffeomorphism invariance) Mukohyama & Noui (2019)

  $\longrightarrow$ minimally-modified gravity $\supset$ extended cuscuton

- As an example, consider the following:

$$S = S_{\mathrm{EH}} + \int \mathrm{d}^4x\,\sqrt{-g}\left(-M_L^2\sqrt{2X} - V(\phi) - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi\right) + \int \mathrm{d}^4x\,\sqrt{-g}\lambda\left[-\frac{3\lambda}{M_{\mathrm{pl}}^2}(2X) + \ln\left(\frac{2X}{\Lambda^4}\right)\Box\phi\right]$$

- FRW (pick $\dot\phi > 0$):

$$3M_{\mathrm{pl}}^2\Theta^2 = \frac{1}{2}\dot\chi^2 + V(\phi)$$
$$2M_{\mathrm{pl}}^2\dot\Theta = -\dot\chi^2 + (M_L^2 + 6\lambda\Theta)\dot\phi$$
$$3M_L^2\Theta = V_{,\phi} - \frac{6\lambda}{M_{\mathrm{pl}}^2}V(\phi)$$

where

$$\Theta \equiv H + \frac{\lambda}{M_{\mathrm{pl}}^2}\dot\phi$$

Cosmological perturbations

- Consider the spatially-flat gauge. The solution to the set of perturbed Hamiltonian and momentum constraints read ($M_{\mathrm{pl}} = 1$)

$$
\begin{aligned}
\Phi^S &= \frac{1}{2\Theta}\left(\dot{\chi}\delta\chi^S - (M_L^2 + 6\lambda\Theta)\delta\phi^S + 2\lambda\dot{\delta\phi}^S\right)\,, \\
aB^S &= -\frac{\lambda}{\Theta}\delta\phi^S + \frac{a^2}{2k^2\Theta^2}\Big[\dot{\chi}\Big((3\Theta^2 - \frac{\dot{\chi}^2}{2})\delta\chi^S + \Theta\dot{\delta\chi}^S\Big) \\
&\quad + \frac{\dot{\chi}^2}{2}\left(M_L^2\delta\phi^S - 2\lambda\dot{\delta\phi}^S\right)\Big]
\end{aligned}
$$

$\longrightarrow$ potentially dangerous when $\Theta = 0$

- With

$$\zeta \equiv -\frac{\Theta}{\dot{\chi}}\delta\chi^S + \lambda\delta\phi^S\,,$$

one finds

$$S_{\mathrm{s}}^{(2)} = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3\mathbf{k}\,az^2\left(\dot{\zeta}_k^2 - c_{\mathrm{s}}^2\frac{k^2}{a^2}\zeta_k^2\right)$$

where

$$z^2 = \frac{a^2\dot{\chi}^2}{\Theta^2 + \frac{M_L^4\frac{\dot{\chi}^2}{2}(\frac{\dot{\chi}^2}{2}+\dot{\Theta})}{(M_L^2+6\lambda\Theta)\left((M_L^2+8\lambda\Theta)k^2/a^2+3M_L^2\frac{\dot{\chi}^2}{2}\right)}} > 0\,,$$

$$c_{\mathrm{s}}^2 = \frac{\tilde{A}_4(k/a)^4 + \tilde{A}_2(k/a)^2 + \tilde{A}_0}{\tilde{B}_4(k/a)^4 + \tilde{B}_2(k/a)^2 + \tilde{B}_0} = 1 + \mathcal{O}\left(\frac{a^2}{k^2}\right) > 0$$

What happens when $\Theta = 0$?

- Apparent divergences actually exactly cancel out!

$$\delta\phi^S = \frac{\zeta}{\lambda} + \mathcal{O}(\Theta)$$

$$\implies \delta\chi^S = -\frac{\dot{\chi}}{\Theta}\zeta + \lambda\frac{\dot{\chi}}{\Theta}\delta\phi^S = \mathcal{O}(\Theta^0)$$

$$= \frac{\dot{\chi}}{2\lambda(\dot{\chi}^2/2 + \dot{\Theta})}(M_L^2\zeta - 2\lambda\dot{\zeta}) + \mathcal{O}(\Theta)$$

- Similarly,

$$\Phi^S = \mathcal{O}(\Theta^0) \qquad \text{and} \qquad aB^S = \mathcal{O}(\Theta^0)$$

Sound speed near the bounce