Limiting Curvature in the Very Early Universe

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Based on
JQ & Yoshida, JCAP02(2020)016, arXiv:1911.06040
Sakakihara, Yoshida, Takahashi & JQ, arXiv:2005.10844

How can we avoid a singularity?

- GR + effective matter satisfying the null energy condition (NEC)

  $\Longrightarrow$ inevitable singularities singularity theorems by Penrose and Hawking
- Even inflationary cosmology (within GR) is inevitably past incomplete and often inextendible Borde & Vilenkin [gr-qc/9312022], Border et al. [gr-qc/0110012], Yoshida & JQ [1803.07085], ...
- Singularity resolution, **before inflation or in alternatives**

  $\longrightarrow$ need to violate the NEC, with e.g.:
  - quantum fields
  - modified gravity
  - full quantum gravity
- Why is this *not* too crazy? E.g.,
  - traversable wormholes Maldacena et al. [1807.04726], ...
  - 'averaged' energy conditions, e.g. Freivogel & Krommydas [1807.03808]

$$\langle T_{\mu\nu}k^\mu k^\nu\rangle_\tau \geq -\frac{\mathcal{O}(1)}{G_\mathrm{N}\tau^2}$$

  - α' corrections in string theory
  - minimal fundamental length in quantum gravity Hossenfelder [1203.6191]
  - etc.

How hard can it be?

- It's kind of difficult...
- A popular avenue: consider a generic scalar-tensor theory, e.g., **Horndeski**, with many free functions $\longrightarrow$ those admit non-singular cosmological background solutions
- However, perturbations are often plagued with **ghosts and gradient instabilities** $\longrightarrow$ indications of a no-go theorem Libanov et al. [1605.05992], Kobayashi [1606.05831], Creminelli et al. [1610.04207], Cai et al. [1610.03400], ...
- Very few ways of evading the no-go theorem and often at some costs, e.g., **strong coupling** issues Ijjas & Steinhardt [1606.08880,1609.01253], Cai et al. [1701.04330], Cai & Piao [1705.03401], Kolevator et al. [1705.06626], Dobre et al. [1712.10272], Mironov et al. [1807.08361,1905.06249,1910.07019], Banerjee et al. [1808.01170], Ye & Piao [1901.08283], ...

Different approach to singularity resolution

- Impose constraint equations that ensure the boundedness of curvature
  $\Longrightarrow$ **limiting curvature**

- Example of implementation Mukhanov & Brandenberger ['92], Brandenberger et al. [gr-qc/9303001], ...

$$S = S_{\mathrm{EH}} + \int \mathrm{d}^4x\,\sqrt{-g}\left[\sum_{i=1}^{n}\chi_i\mathcal{I}_i(\mathbf{Riem},\boldsymbol{g},\boldsymbol{\nabla}) - V(\chi_1,...,\chi_n)\right]$$

$$\delta_{\chi_i}S = 0 \implies \mathcal{I}_i = \partial_{\chi_i}V$$

$$|\partial_{\chi_i}V| < \infty\ \forall\chi_i \implies \text{bounded curvature } \mathcal{I}_i$$

- Concrete model (e.g., $n = 2$)

$$\mathcal{I}_1 = \sqrt{12R_{\mu\nu}R^{\mu\nu} - 3R^2} \overset{\mathrm{FLRW}}{\propto} \dot{H}\,, \qquad \mathcal{I}_2 = R + \mathcal{I}_1 \overset{\mathrm{FLRW}}{\propto} H^2$$

$\longrightarrow$ non-singular background cosmology, but severe instabilities
Yoshida, JQ, Yamaguchi, Brandenberger [1704.04184]

- Another implementation of limiting curvature: mimetic gravity

  Chamseddine & Mukhanov [1308.5410,1612.05860], ...

$$S = S_{\rm EH} + \int \mathrm{d}^4x\,\sqrt{-g}\,[\lambda(\partial_\mu\phi\partial^\mu\phi + 1) + \chi\Box\phi - V(\chi)]$$

$$\delta_\lambda S = 0 \implies \partial_\mu\phi\partial^\mu\phi = -1$$

$$\delta_\chi S = 0 \implies \Box\phi = \partial_\chi V \longrightarrow \chi_{\rm sol} = \chi(\Box\phi)$$

$$f(\Box\phi) = \chi_{\rm sol}\Box\phi - V(\chi_{\rm sol}) \longrightarrow \mathcal{L}_\phi = \lambda(\partial_\mu\phi\partial^\mu\phi + 1) + f(\Box\phi)$$

- E.g. in FLRW, $\phi = t \implies \Box\phi = 3H$, so bounding $\partial_\chi V$ ensures H does not blow up

- Yet, mimetic gravity suffers from (gradient) instabilities Ijjas et al. [1604.08586], Firouzjahi et al. [1703.02923], Takahashi & Kobayashi [1708.02951], Langlois et al. [1802.03394], ...

- Also, anisotropies can still blow up beyond the FLRW approximation

  de Cesare et al. [2002.11658]

Cuscuton gravity

- Setup: GR + non-dynamical scalar field ϕ on cosmological background
- Original implementation: start with k-essence theory
  Afshordi et al. [hep-th/0609150], ...

$$S = S_{\rm EH} + \int \mathrm{d}^4x\,\sqrt{-g}P(X,\phi)\,, \qquad X \equiv -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$$

$$\delta_\phi S = 0 \overset{\rm FLRW}{\Longrightarrow} (P_{,X} + 2XP_{,XX})\ddot{\phi} + 3HP_{,X}\dot{\phi} + P_{,X\phi}\dot{\phi}^2 - P_{,\phi} = 0$$

- Requiring $P_{,X} + 2XP_{,XX} = 0$ sets

$$P(X,\phi) = c_1(\phi)\sqrt{X} + c_2(\phi)$$

- Rescaling ϕ, we can write

$$\mathcal{L}_{\rm cuscuton} = \pm M_L^2\sqrt{2X} - V(\phi)\,, \qquad \partial_\mu\phi \text{ timelike}$$

- EOM becomes a constraint equation:

$$\mp\mathrm{sgn}(\dot{\phi})3M_L^2H = \partial_\phi V$$

$\longrightarrow$ limiting extrinsic curvature

$$M_L^2K = \partial_\phi V\,, \qquad K = \nabla_\mu u^\mu\,, \qquad u_\mu = \pm\frac{\partial_\mu\phi}{\sqrt{2X}}$$

$\longrightarrow$ non-singular bouncing models Boruah et al. [1802.06818]

- Cuscuton fluctuations do not propagate:

$$S^{(2)}_{\mathrm{scalar}} = \int \mathrm{d}^3x\mathrm{d}t\, a^3\left(\mathcal{G}\dot{\zeta}^2 - \frac{\mathcal{F}}{a^2}(\vec{\nabla}\zeta)^2\right)\,,$$

$$\mathcal{G} = \frac{X}{H^2}(P_{,X} + 2XP_{,XX}) = 0\,, \qquad \mathcal{F} = -M_{\mathrm{pl}}^2\dot{H}/H^2$$

- Interesting properties:
  - forms no caustics de Rham & Motohashi [1611.05038]
  - geometrical interpretation Chagoya & Tasinato [1610.07980]
  - new symmetries Pajer & Stefanyszyn [1812.05133], Grall et al. [1909.04622]

New unifying approach: limiting extrinsic curvature

Sakakihara, Yoshida, Takahaski & JQ [2005.10844]

$$S = S_{\mathrm{EH}} + \int_{\Sigma_t} \mathrm{d}^3x \int \mathrm{d}t\, N\sqrt{-\gamma}\left[\sum_{i=1}^{n} \chi_i \mathcal{I}_i(\boldsymbol{K}, \boldsymbol{\gamma}, \boldsymbol{D}) - V(\chi_1, ..., \chi_n)\right]$$

- Characterize the foliation Σ_t with unit normal vector n^μ by a new field:

$$n_\mu = \begin{cases} -\nabla_\mu\phi & \text{with } \nabla_\mu\phi\nabla^\mu\phi = -1 \\ A_\mu & \text{with } A_\mu A^\mu = -1 \end{cases}$$

- One extrinsic curvature invariant, $\mathcal{I}_1 = K = \nabla^\mu n_\mu = 3H$ (FLRW):

$$\mathcal{L} = \mathcal{L}_{\mathrm{EH}} + \lambda(\nabla_\mu\phi\nabla^\mu\phi + 1) - \chi\nabla^\mu\nabla_\mu\phi - V(\chi) \longrightarrow \text{mimetic}$$
$$\mathcal{L} = \mathcal{L}_{\mathrm{EH}} + \lambda(A_\mu A^\mu + 1) + \chi\nabla^\mu A_\mu - V(\chi) \longrightarrow \text{cuscuton}$$

- Mimetic: Legendre transformation $f(\Box\phi) = -\chi\Box\phi - V(\chi)$
- Cuscuton: EOM $A_\mu = \nabla_\mu\chi/2\lambda \implies \lambda = \pm(-\nabla_\mu\chi\nabla^\mu\chi)^{1/2}/2$ and $A_\mu = \pm\nabla_\mu\chi/\sqrt{-\nabla_\nu\chi\nabla^\nu\chi}$, so

$$\mathcal{L} = \mathcal{L}_{\mathrm{EH}} \pm \sqrt{-\nabla_\mu\chi\nabla^\mu\chi} - V(\chi)$$

Difference between mimetic gravity and the cuscuton

- Cuscuton A_μ EOM:

$$2\lambda A_\mu - \nabla_\mu\chi = 0 \implies \lambda = -\frac{1}{2}A_\mu\nabla^\mu\chi$$

- Mimetic ϕ EOM:

$$\nabla_\mu(2\lambda\nabla^\mu\phi + \nabla^\mu\chi) = 0 \implies \lambda = \frac{1}{2}\nabla_\mu\phi(\nabla^\mu\chi + U^\mu)\,, \text{ with } \nabla_\mu U^\mu = 0$$

- In FLRW: $\nabla_\mu U^\mu = \dot{U}^0 + 3HU^0 = 0 \implies U^0 \propto a^{-3}$
  and $\rho_u \propto U^0 \propto a^{-3} \implies$ dust (mimetic dark matter)

- Cuscuton has one fewer d.o.f. than mimetic $\implies$ often more stable

Example: cuscuton with matter

$$\mathcal{L} = \mathcal{L}_{\mathrm{EH}} - M_L^2\sqrt{-\partial_\mu\phi\partial^\mu\phi} - V(\phi) - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi$$

$$\overset{\mathrm{FLRW}}{\Longrightarrow} \left(2M_{\mathrm{pl}}^2 - \frac{3M_L^4}{V_{,\phi\phi}}\right)\dot{H} = -\dot{\chi}^2\,, \qquad 0 < V_{,\phi\phi} < \frac{3}{2}\frac{M_L^4}{M_{\mathrm{pl}}^2} \implies \dot{H} > 0$$

- No ghost and no gradient instability Boruah et al. [1802.06818], JQ & Yoshida [1911.06040]
- Interesting behaviour near $H \approx 0$ (bounce):

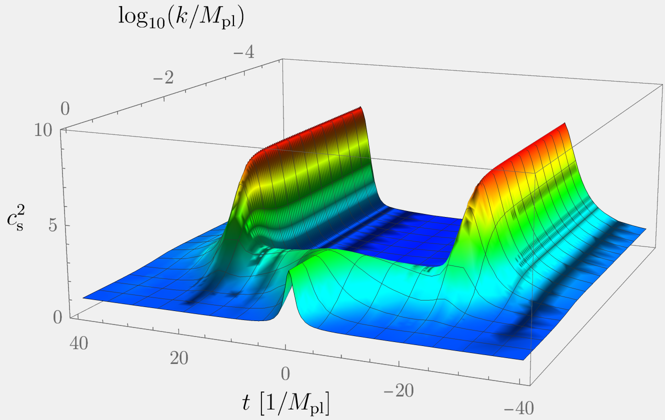
$$S^{(2)}_{\mathrm{scalar}} \overset{k\to\infty}{\simeq} \frac{4}{M_L^2}\int \mathrm{d}^3k\mathrm{d}t\,\frac{ak^2}{|\dot{\phi}|}\left[\dot{\zeta}_k^2 - \left(1 + \frac{\dot{H}}{\dot{\chi}^2}\right)\frac{k^2}{a^2}\zeta_k^2\right]$$

- Defining $m^2 \equiv V_{,\phi\phi}|_{\mathrm{bounce}}$:

$$c_{\mathrm{s}}^2 \overset{\frac{k}{a}\ll\mathcal{O}(\dot{\chi})}{\sim} -\frac{1}{3} + \frac{4m^2M_{\mathrm{pl}}^2}{3(3M_L^4 - 2m^2M_{\mathrm{pl}}^2)} \in (0,1] \quad \text{if} \quad \frac{1}{2} < \frac{m^2M_{\mathrm{pl}}^2}{M_L^4} \leq 1$$

$$c_{\mathrm{s}}^2 \overset{\frac{k}{a}\gtrsim\mathcal{O}(\dot{\chi})}{\sim} 1 + \frac{4m^2M_{\mathrm{pl}}^2}{3M_L^4 - 2m^2M_{\mathrm{pl}}^2} \sim \mathcal{O}(1-10)$$

Sound speed



- $c_s^2 > 1$, but causality remains fine Bruneton [gr-qc/0607055], Babichev, Mukhanov, Vikman [0708.0561], de Rham & Tolley [2007.01847]
- Safe from strong coupling?

Other application: bounded anisotropies

- Let's come back to

$$\mathcal{L} \supset \sum_{i=1}^{n} \chi_i \mathcal{I}_i(\boldsymbol{K}, \boldsymbol{\gamma}, \boldsymbol{D}) - V(\chi_1, ..., \chi_n)$$

and consider $\mathcal{I}_1 = K^2$ and $\mathcal{I}_2 = K^\mu{}_\nu K^\nu{}_\mu - \frac{1}{3}K^2$

- With $n_\mu = A_\mu$ (cuscuton-like)

$$\mathcal{L} \supset \chi_1 (\nabla^\mu A_\mu)^2 + \chi_2 \left(\nabla^\mu A_\nu \nabla^\nu A_\mu - \frac{1}{3} (\nabla^\mu A_\mu)^2 \right) - V(\chi_1, \chi_2)$$

- In a Bianchi I spacetime

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2 \left(e^{2\beta_+ + 2\sqrt{3}\beta_-} \mathrm{d}x^2 + e^{2\beta_+ - 2\sqrt{3}\beta_-} \mathrm{d}y^2 + e^{-4\beta_+} \mathrm{d}z^2 \right)$$

we then have $\mathcal{I}_1 = 9H^2 = \partial_{\chi_1} V$ and

$$\mathcal{I}_2 = 6\Sigma^2 \equiv 6 \left(\dot{\beta}_+^2 + \dot{\beta}_-^2 \right) = \partial_{\chi_2} V \longrightarrow \text{can bound anisotropies}$$

Toy model

- Consider vacuum. Anisotropy EOM:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[a^3(1+2\chi_2)\dot{\beta}_\pm\right]=0 \implies \rho_{\mathrm{ani}}\propto\Sigma^2=\frac{\Sigma_0^2}{(1+2\chi_2)^2 a^6}$$

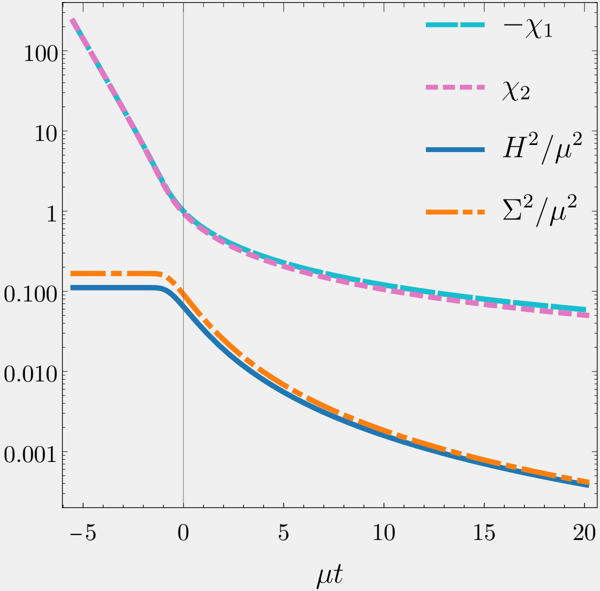
- $\rho_{\mathrm{ani}}\to\mathrm{const.}$ as $\chi_2\sim a^{-3}$ at early times

- E.g.: $V(\chi_1,\chi_2)=\mu^2(\chi_1-\tanh\chi_1+\chi_2-\tanh\chi_2)$

  $\implies H^2\to\mu^2/9$ and $\Sigma^2\to\mu^2/6$ at early times, GR at late times

$$\mathrm{d}s^2 \overset{t\to-\infty}{\simeq} -\mathrm{d}t^2+\sum_{i=1}^{3}e^{2H_i t}(\mathrm{d}x^i)^2$$

$$H_x=H_y=\left(\frac{1}{3}\pm\frac{1}{\sqrt{6}}\right)\mu\,,\quad H_z=\left(\frac{1}{3}\mp\frac{2}{\sqrt{6}}\right)\mu$$



$-\chi_1$
χ_2
H^2/μ^2
Σ^2/μ^2
100
10
1
0.100
0.010
0.001
−5
0
5
10
15
20
μt

Perturbations

- Only 2 d.o.f. (like the polarization states of GWs in FLRW)
- Consider $\beta_- = 0$ (so $\beta_+ \equiv \beta$) and $k_i \mathrm{d}x^i = k_y \mathrm{d}y + k_z \mathrm{d}z$ by rotational symmetry
- Vector perturbations:

$$\delta g_{\mu\nu} = \begin{pmatrix} 0 & \delta E & 0 & 0 \\ * & 0 & -a^2 e^{2\beta} \partial_z h_\times & a^2 e^{-4\beta} \partial_y h_\times \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix}, \qquad \delta A_\mu = (0, \delta A_x, 0, 0)$$

$$\mathcal{L}_{\mathrm{V}}^{(2)} = \frac{M_{\mathrm{pl}}^2}{2} k^2 a^3 e^{-4\beta} (1+2\chi_2) \left[\dot{h}_{\times,-\mathbf{k}} \dot{h}_{\times,\mathbf{k}} - \left(\frac{k^2}{(1+2\chi_2)a^2} + 36 \frac{e^{2\beta} k_y^2 k_z^2}{k^4} \sigma^2 \right) h_{\times,-\mathbf{k}} h_{\times,\mathbf{k}} \right]$$

- No ghost and no gradient instability as long as

$$1 + 2\chi_2 > 0$$

✓ $\chi_2 \geq 0$ in the example earlier ($\chi_2 \to 0$ in the late-time, GR limit)

- Scalar perturbations:

$$\delta g_{\mu\nu} = \begin{pmatrix} -2\Phi & 0 & a(\partial_y B + e^{2\beta}\partial_z S) & a(\partial_z B - e^{-4\beta}\partial_y S) \\ 0 & -a^2(\partial_y^2 + e^{6\beta}\partial_z^2)h_+ & 0 & 0 \\ * & 0 & a^2 e^{6\beta}\partial_z^2 h_+ & -a^2\partial_y\partial_z h_+ \\ * & 0 & * & a^2 e^{-6\beta}\partial_y^2 h_+ \end{pmatrix}$$

$$\delta A_\mu = (\delta A_0, 0, \partial_y \delta A, \delta A_z)\,, \qquad \delta\lambda\,, \quad \delta\chi_1\,, \quad \delta\chi_2$$

$$\mathcal{L}_{\mathrm{S}}^{(2)} = \frac{M_{\mathrm{pl}}^2}{2} a^3 k^4 (1+2\chi_2)\left[\mathcal{G}(k_y,k_z)\dot{h}_{+,\mathbf{k}}\dot{h}_{+,-\mathbf{k}} - \frac{k^2}{(1+2\chi_2)a^2} h_{+,\mathbf{k}} h_{+,-\mathbf{k}}\right]$$

✓ Gradient instabilities are avoided when $1+2\chi_2 > 0$

✗ $\mathcal{G}(k_y,k_z) > 0$ only if $k_y/k_z \sim \mathcal{O}(0.1-10)$; ghost mode otherwise

So what have we learned

- Wide class of spatially-covariant theories (unifying framework for scalar-tensor theories of gravity) Gao [1406.0822], ...
- A subclass of those can be nicely written as **limiting extrinsic curvature theories**
- *Mimetic* gravity and the *Cuscuton* are such theories
- Those admit non-singular FLRW and even Bianchi I cosmologies

So what have we learned

- Cuscuton $\implies$ fully stable (classical, linear); evades the no-go!
- $c_{\mathrm{s}}^2 \sim 1$, but superluminality close to the bounce; is it a valid EFT?
- Generalizes to the extended cuscuton $\longrightarrow$ stable bounce
  (additional slides)
- We are in order to understand the evolution of cosmological perturbations through a bounce
  $\longrightarrow$ upper bound on the growth of IR perturbations
  (additional slides)

- Non-singular Bianchi I toy model in vacuum
- Not fully stable; can it be improved?

Thank you for your attention!

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Additional slides

Extended cuscuton

- Rather than starting with $P(X, \phi)$, start with Horndeski or even beyond-Horndeski theory, and impose Iyonaga et al. [1809.10935]
  1. the background EOM to be at most a first-order constraint equation
  2. and the kinetic term of scalar perturbations to vanish

  $\longrightarrow$ extended cuscuton $\supset$ original cuscuton

- Alternatively, in the ADM formalism, one can construct a Hamiltonian, satisfying the appropriate conditions for the theory to propagate at most 2 gravitational d.o.f. and remaining invariant under 3-D diffeomorphisms (but possibly breaking time diffeomorphism invariance) Mukohyama & Noui [1905.02000]

  $\longrightarrow$ minimally-modified gravity $\supset$ extended cuscuton

- As an example, consider the following:

$$S = S_{\rm EH} + \int \mathrm{d}^4x\,\sqrt{-g}\left(-M_L^2\sqrt{2X} - V(\phi) - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi\right)$$
$$+ \int \mathrm{d}^4x\,\sqrt{-g}\lambda\left[-\frac{3\lambda}{M_{\rm pl}^2}(2X) + \ln\left(\frac{2X}{\Lambda^4}\right)\Box\phi\right]$$

- FLRW (pick $\dot{\phi} > 0$):

$$3M_{\rm pl}^2\Theta^2 = \frac{1}{2}\dot{\chi}^2 + V(\phi)$$
$$2M_{\rm pl}^2\dot{\Theta} = -\dot{\chi}^2 + (M_L^2 + 6\lambda\Theta)\dot{\phi}$$
$$3M_L^2\Theta = V_{,\phi} - \frac{6\lambda}{M_{\rm pl}^2}V(\phi)$$

where

$$\Theta \equiv H + \frac{\lambda}{M_{\rm pl}^2}\dot{\phi}$$

Cosmological perturbations

- With (spatially-flat gauge)

$$\zeta \equiv -\frac{\Theta}{\dot{\chi}}\delta\chi^S + \lambda\delta\phi^S$$

one finds

$$S_{\mathrm{s}}^{(2)} = \frac{1}{2}\int \mathrm{d}^3k\mathrm{d}t\, az^2 \left(\dot{\zeta}_k^2 - c_{\mathrm{s}}^2\frac{k^2}{a^2}\zeta_k^2\right)$$

where

$$z^2 = \frac{a^2\dot{\chi}^2}{\Theta^2 + \frac{M_L^4\frac{\dot{\chi}^2}{2}(\frac{\dot{\chi}^2}{2}+\dot{\Theta})}{(M_L^2+6\lambda\Theta)\left((M_L^2+8\lambda\Theta)k^2/a^2+3M_L^2\frac{\dot{\chi}^2}{2}\right)}} > 0 \qquad \checkmark$$

$$c_{\mathrm{s}}^2 = \frac{\tilde{A}_4(k/a)^4 + \tilde{A}_2(k/a)^2 + \tilde{A}_0}{\tilde{B}_4(k/a)^4 + \tilde{B}_2(k/a)^2 + \tilde{B}_0} = 1 + \mathcal{O}\left(\frac{a^2}{k^2}\right) > 0 \qquad \checkmark$$

Cuscuton gravity with matter

- Consider the addition of a massless scalar field

$$\mathcal{L} = \mathcal{L}_{\rm EH} \pm M_L^2\sqrt{2X} - V(\phi) - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi$$

$$\overset{\rm FRW}{\Longrightarrow}\ 3M_{\rm pl}^2H^2 = \frac{1}{2}\dot{\chi}^2 + V(\phi)\,, \quad 2M_{\rm pl}^2\dot{H} = -\dot{\chi}^2 \mp M_L^2|\dot{\phi}|$$

- Choose '$-$' sign in $\mathcal{L}_{\rm cuscuton}$
- NEC violation:

$$M_L^2|\dot{\phi}| > \dot{\chi}^2 \implies 2M_{\rm pl}^2\dot{H} = -\dot{\chi}^2 + M_L^2|\dot{\phi}| > 0$$

- Requirement for a bounce:

$$\mathrm{sgn}(\dot{\phi})3M_L^2H = V_{,\phi} \implies 3M_L^2\dot{H} = V_{,\phi\phi}|\dot{\phi}|$$
$$V_{,\phi\phi} > 0 \implies \dot{H} > 0$$

Cosmological perturbations

- Consider the comoving gauge w.r.t. ϕ, so $\delta\phi = 0$, but $\chi(t, \mathbf{x}) = \chi(t) + \delta\chi(t, \mathbf{x})$ and

$$\mathrm{d}s^2 = -(1 + 2\Phi)\mathrm{d}t^2 + 2a\partial_i B \mathrm{d}x^i \mathrm{d}t + a^2(1 - 2\Psi)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j$$

- Perturbed Hamiltonian and momentum constraints in Fourier space (setting $M_{\mathrm{pl}} = 1$):

$$(\dot{\chi}^2/2 - 3H^2)\Phi_k + H(k/a)^2 B_k + 3H\dot{\Psi}_k + (k/a)^2\Psi_k - \dot{\chi}\delta\dot{\chi}_k = 0$$
$$2H\Phi_k - 2\dot{\Psi}_k - \dot{\chi}\delta\chi_k = 0$$

- $\longrightarrow$ need to divide by H (in particular when $H = 0$) to eliminate Φ_k and B_k
$\longrightarrow$ potential divergences

- After simplification,

$$S^{(2)}_{\rm scalar}=\int \mathrm{d}^3k\mathrm{d}t\, az^2\left(\dot\zeta_k^2-c_{\rm s}^2\frac{k^2}{a^2}\zeta_k^2\right)\,,\qquad \zeta_k=-\Psi_k-\frac{H}{\dot\chi}\delta\chi_k\,,$$

where

$$z^2=a^2\frac{\dot\chi^2(k^2/a^2+3\dot\chi^2/2)}{(k/a)^2H^2+\dot\chi^2(3H^2+\underbrace{\dot H+\dot\chi^2/2}_{=M_L^2|\dot\phi|/2})/2}>0\,,\quad \checkmark$$

$$c_{\rm s}^2=\frac{H^4k^4/a^4+A_2k^2/a^2+A_0}{H^4k^4/a^4+B_2k^2/a^2+B_0}\xrightarrow{k\to\infty}1>0\,,\quad\checkmark$$

with

$$A_2\equiv\dot\chi^2/2\left(12H^2+3\dot H+\dot\chi^2/2\right)+2\dot H^2-H\ddot H$$

$$A_0\equiv\left(\dot\chi^2/2\right)^2\left(15H^2+\dot H-\dot\chi^2/2\right)-\dot\chi^2/2\left(12H^2\dot H-2\dot H^2+3H\ddot H\right)$$

$$B_2\equiv\dot\chi^2/2\left(6H^2+\dot H+\dot\chi^2/2\right)\,,\ B_0\equiv3\left(\dot\chi^2/2\right)^2\left(3H^2+\dot H+\dot\chi^2/2\right)$$

- Note, however,

$$z^2\xrightarrow{k\to\infty}a^2\dot\chi^2/H^2\xrightarrow{H\to0}\infty$$

Switch gauge

- Spatially flat ($\Psi^S = 0$):

$$\Phi_k^S = -\frac{\mathrm{d}}{\mathrm{d}t}(\zeta_k/H) + \mathcal{O}(H^0)\,,\ aB_k^S = \zeta_k/H + \mathcal{O}(H^0)\,,$$
$$\delta\chi_k^S = -\dot{\chi}\zeta_k/H + \mathcal{O}(H^0)\,,\ \delta\phi_k^S = -\dot{\phi}\zeta_k/H + \mathcal{O}(H^0)$$
$$\implies \text{ill defined at } H = 0$$

- Back to comoving gauge w.r.t. ϕ ($\delta\phi^\phi = 0$):

$$\Phi_k^\phi = \Phi_k^S - \frac{\mathrm{d}}{\mathrm{d}t}(\delta\phi_k^S/\dot{\phi}) = -\frac{4}{1 + 3\dot{\chi}^2 a^2/2k^2}\zeta_k + \mathcal{O}(H)$$
$$aB_k^\phi = aB_k^S + \delta\phi_k^S/\dot{\phi} = -\frac{3a^2\dot{\chi}^2}{M_L^2 k^2 \dot{\phi}}\dot{\zeta}_k + \mathcal{O}(H)$$
$$\Psi_k^\phi = H\delta\phi_k^S/\dot{\phi} = -\zeta_k + \mathcal{O}(H)$$
$$\delta\chi_k^\phi = \delta\chi_k^S - \dot{\chi}\delta\phi_k^S/\dot{\phi} = -\frac{2\dot{\chi}}{M_L^2\dot{\phi}}\dot{\zeta}_k + \mathcal{O}(H)$$

- $\Longrightarrow$ divergences exactly cancel out to yield well-defined perturbations at $H = 0$

  $\Longrightarrow$ valid perturbed action $\mathcal{L}_\mathrm{s}^{(2)} = az^2(\dot{\zeta}_k^2 - c_\mathrm{s}^2 k^2 \zeta_k^2/a^2)$

- Comoving gauge w.r.t. χ ($\delta\chi^\chi = 0$):

$$\Phi_k^\chi = \Phi_k^S - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta\chi_k^S}{\dot{\chi}}\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\zeta_k}{H}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\zeta_k}{H}\right) + \mathcal{O}(H^0)$$

$$aB_k^\chi = aB_k^S + \frac{\delta\chi_k^S}{\dot{\chi}} = \frac{\zeta_k}{H} + \left(-\frac{\zeta_k}{H}\right) + \mathcal{O}(H^0)$$

$$\Psi_k^\chi = H\frac{\delta\chi_k^S}{\dot{\chi}} = H\left(-\frac{\zeta_k}{H}\right) + \mathcal{O}(H^0)$$

$$\delta\phi_k^\chi = \delta\phi_k^S - \dot{\phi}\frac{\delta\chi_k^S}{\dot{\chi}} = -\frac{\dot{\phi}}{H}\zeta_k - \dot{\phi}\left(-\frac{\zeta_k}{H}\right) + \mathcal{O}(H^0)$$

$\longrightarrow$ all finite at $H = 0$

- Newtonian gauge ($B^N = 0$):

$$\Phi_k^N = \Phi_k^S + \frac{\mathrm{d}}{\mathrm{d}t}(aB_k^S) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\zeta_k}{H}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\zeta_k}{H}\right) + \mathcal{O}(H^0)$$

$$\Psi_k^N = -aHB_k^S = -H\frac{\zeta_k}{H} + \mathcal{O}(H^0)$$

$$\delta\phi_k^N = \delta\phi_k^S + a\dot{\phi}B_k^S = -\dot{\phi}\frac{\zeta_k}{H} + \dot{\phi}\frac{\zeta_k}{H} + \mathcal{O}(H^0)$$

$$\delta\chi_k^N = \delta\chi_k^S + a\dot{\chi}B_k^S = -\dot{\chi}\frac{\zeta_k}{H} + \dot{\chi}\frac{\zeta_k}{H} + \mathcal{O}(H^0)$$

$$\longrightarrow \text{all finite at } H = 0$$

Evolution of ζ_k in the IR in a bounce phase

- The evolution of ζ_k in the IR through a bounce phase links perturbations from a contracting phase (scale invariant?) to the CMB
- For $k \to 0$,

$$\ddot{\zeta} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{z}}{z}\right)\dot{\zeta} = 0 \implies \zeta = \mathrm{const.} \ \text{ and } \ \zeta(t) \propto \int^t \frac{\mathrm{d}t}{az^2}$$

- Can ζ undergo significant amplification? Generally not the case, but if so, possibly important non-Gaussianities generated Battarra et al. [1404.5067], JQ et al. [1508.04141]
- In general, if $z \propto a$ (constant EoS), then $\Delta\zeta < \dot{\zeta}_i (a_i/a_B)^3 \Delta t$
- Here,

$$z^2 \overset{k\to 0}{\simeq} \frac{3a^2\dot{\chi}^2/M_{\mathrm{pl}}^2}{3H^2 + \dot{H} + \dot{\chi}^2/2M_{\mathrm{pl}}^2} \not\propto a^2$$

- One finds $\Delta\zeta < \dot{\zeta}_i(a_i/a_B)^3\mathcal{E}\Delta t$ with

$$\mathcal{E} = \frac{1+3\left(1-\frac{3}{2}\frac{M_L^4}{m^2M_{\rm pl}^2}\right)\left(\frac{a_i}{a_B}\right)^3\left(\frac{H_i^2}{\dot{H}_B}+\frac{1}{3}\right)}{1+3\left(1-\frac{3}{2}\frac{M_L^4}{m^2M_{\rm pl}^2}\right)\left(\frac{a_i}{a_B}\right)^6\frac{H_i^2}{\dot{H}_B}}$$

- Recall

$$1-\frac{3}{2}\frac{M_L^4}{m^2M_{\rm pl}^2}\sim\mathcal{O}(1)\,,\qquad \text{so }\mathcal{E}\gg 1\text{ is impossible}$$

- $\longrightarrow$ large wavelength curvature perturbations passing through a bounce cannot receive more amplification than $\mathcal{O}(\dot{\zeta}_i(a_i/a_B)^3\Delta t)$