Quantum Circuit Complexity of Primordial Perturbations

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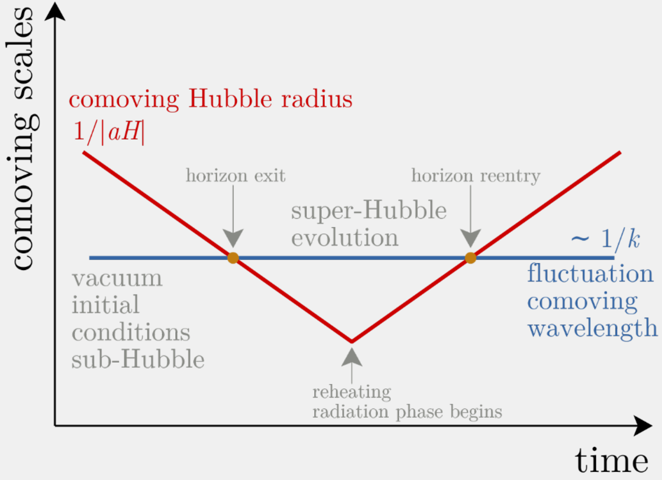
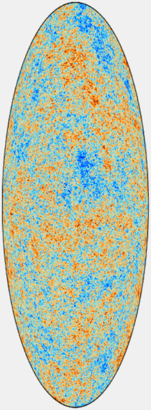
Based on
Jean-Luc Lehners & JQ, arXiv:2012.04911

Outline

1. Motivation
2. Introduction to quantum circuit complexity
3. Review of the quantum-to-classical transition
   of very early universe cosmological perturbations
4. Results and conclusions!

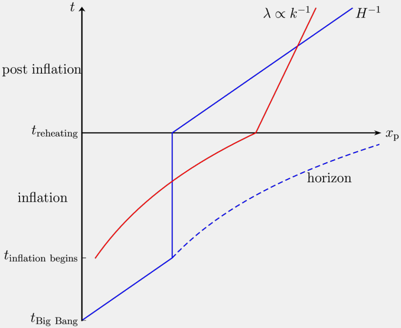
How do we get the CMB?

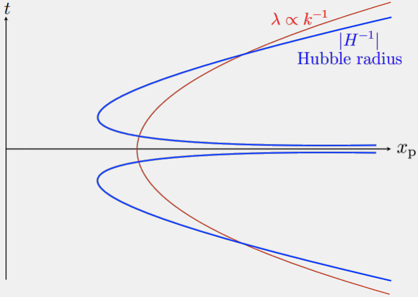




A few possibilities for the early universe

Inflationary, bouncing, emerging, ...

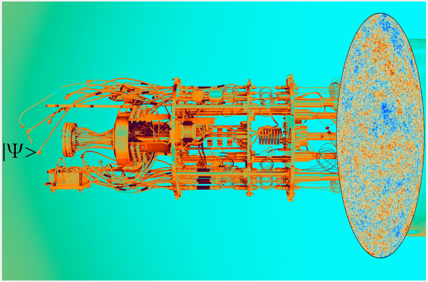


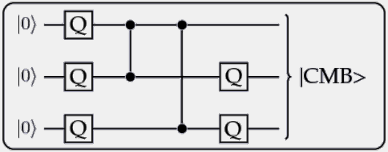


$\longrightarrow$ various models can explain the CMB with more or less success

The early universe as a quantum computer

- If we were to simulate the perturbations of the early universe with a quantum computer, **how complex** would it be?
- With a quantum circuit, **how many quantum gates** would it require?





$\longrightarrow$ actual quantum algorithms can be constructed!

e.g., Li & Liu, "On Quantum Simulation Of Cosmic Inflation" [2009.10921]

An introduction to quantum circuits

e.g., Jefferson & Myers [1707.08570]

- Start with a Reference and Target state, both Gaussian and with respective frequencies ω and Ω (1-d harmonic oscillators):

$$|\Psi_{\mathrm{R}}\rangle = \left(\frac{\omega}{\pi}\right)^{1/4} e^{-\frac{1}{2}\omega x^2}\,, \quad |\Psi_{\mathrm{T}}\rangle = \left(\frac{\Omega}{\pi}\right)^{1/4} e^{-\frac{1}{2}\Omega x^2}\,, \quad |\Psi_{\mathrm{T}}\rangle = \hat{U}|\Psi_{\mathrm{R}}\rangle$$

- Example of gates that can constitute the unitary evolution ($\hat{p}_x = -i\hat{\partial}_x$):

$$\hat{A} \equiv e^{i\epsilon} \qquad \hat{A}|\Psi(x)\rangle = e^{i\epsilon}|\Psi(x)\rangle$$

$$\hat{J} \equiv e^{i\epsilon\hat{p}_x} = e^{\epsilon\hat{\partial}_x} \qquad \hat{J}|\Psi(x)\rangle = |\Psi(x+\epsilon)\rangle$$

$$\hat{Q} \equiv e^{\frac{\epsilon}{2}} e^{i\epsilon\hat{x}\hat{p}_x} = e^{\frac{\epsilon}{2}} e^{\epsilon\hat{x}\hat{\partial}_x} \qquad \hat{Q}|\Psi(x)\rangle = e^{\frac{\epsilon}{2}}|\Psi(e^{\epsilon}x)\rangle$$

- Here take:

$$\hat{U} = \hat{Q}^{\alpha} \quad \longrightarrow \quad \hat{Q}^{\alpha} e^{-\frac{1}{2}\omega x^2} = e^{\frac{\alpha\epsilon}{2}} e^{-\frac{1}{2}\omega e^{2\alpha\epsilon} x^2}$$

$\implies 2\alpha\epsilon = \ln(\Omega/\omega)$ does the job

Circuit complexity

- Complexity counts the # of gates: $\mathcal{C} = \epsilon\alpha = \frac{1}{2}\ln(\Omega/\omega)$
- In cosmology generally $\omega, \Omega \in \mathbb{C}$, so 'analytically continuing' the formula suggests

$$\mathcal{C} = \epsilon|\alpha| = \frac{1}{2}\left|\ln\left(\frac{\Omega}{\omega}\right)\right| = \frac{1}{2}\sqrt{\left(\ln\left|\frac{\Omega}{\omega}\right|\right)^2 + \left(\arctan\left[\frac{\mathrm{Im}(\Omega/\omega)}{\mathrm{Re}(\Omega/\omega)}\right]\right)^2}$$

First done by Bhattacharyya+ [2001.08664,2005.10854]

- The general idea is that a circuit can have a continuous differential-geometry description
  $\Rightarrow$ optimal quantum simulation $\equiv$ smallest number of gates $\equiv$ geodesic

  Nielsen [quant-ph/0502070], Jefferson & Myers [1707.08570], Camargo+ [1807.07075], Ali+ [1810.02734], Chapman+ [1810.05151]

Covariant matrix approach

- Gaussian states $\Rightarrow$ all info in 2-pt functions, which can be split into sym. and anti-sym. parts. Write $\xi^m = (x, p)$:

$$2\langle\Psi|\xi^m\xi_n|\Psi\rangle = G^m{}_n + i\Omega^m{}_n$$

$$\text{cov. matrix :} \qquad G^m{}_n = \langle\Psi|\{\xi^m, \xi_n\}|\Psi\rangle = \begin{pmatrix} 2\langle x^2\rangle & \langle xp+px\rangle \\ \langle xp+px\rangle & 2\langle p^2\rangle \end{pmatrix}$$

$$\text{cano. comm. rel. :} \qquad i\Omega^m{}_n = \langle\Psi|[\xi^m, \xi_n]|\Psi\rangle = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Evolution: $|\Psi_{\mathrm{T}}\rangle = \hat{U}|\Psi_{\mathrm{R}}\rangle \longrightarrow \mathsf{G}_{\mathrm{T}} = \mathsf{U}\mathsf{G}_{\mathrm{R}}\mathsf{U}^{\mathrm{T}}$
- If we write $|\Psi_{\mathrm{R}}\rangle = (k/\pi)^{1/4}e^{-\frac{1}{2}kx^2}$ and $|\Psi_{\mathrm{T}}\rangle = (\mathcal{A}/\pi)^{1/4}e^{-\frac{1}{2}(\mathcal{A}+i\varphi)x^2}$, $k, \mathcal{A}, \varphi \in \mathbb{R}$, then

$$\mathsf{G}_{\mathrm{R}} = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & k \end{pmatrix}, \qquad \mathsf{G}_{\mathrm{T}} = \begin{pmatrix} \frac{1}{\mathcal{A}} & -\frac{\varphi}{\mathcal{A}} \\ -\frac{\varphi}{\mathcal{A}} & \frac{\mathcal{A}^2+\varphi^2}{\mathcal{A}} \end{pmatrix}$$

- Canonical commutation relation preserved under $\xi^m \to \tilde{\xi}^m = M^m{}_n\xi^n$ as long as $\mathsf{M}^{\mathrm{T}}\Omega\mathsf{M} = \Omega \iff \mathsf{M} \in \mathrm{Sp}(2, \mathbb{R})$

Covariant matrix approach

- To go from G_R to G_T, only a sub-algebra of $\mathfrak{sp}(2,\mathbb{R})$ is needed as gates to construct U:

$$\mathsf{M}_1=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\,,\qquad \mathsf{M}_2=\begin{pmatrix}0 & 0\\ \sqrt{2} & 0\end{pmatrix}$$

- Continuous evolution is then

$$\mathsf{U}(s)=\overleftarrow{P}\exp\left(\int_0^s \mathrm{d}\tilde{s}\,Y^I(\tilde{s})\mathsf{M}_I\right)=\begin{pmatrix}\sqrt{z} & 0\\ \frac{y}{\sqrt{2z}} & \frac{1}{\sqrt{z}}\end{pmatrix}$$

where the Y^I's are just on/off switches and (y,z) are rescaled coordinates such that

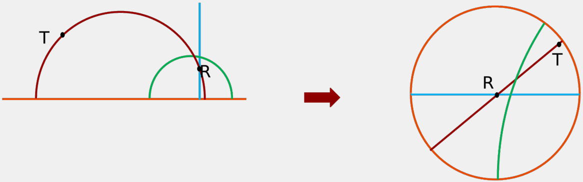
$$(y_0,z_0)=(0,1)\,,\qquad (y,z)=\left(-\frac{\varphi}{\sqrt{2}\mathcal{A}},\frac{k}{\mathcal{A}}\right)$$

- The geometry is found from (choosing a metric $g^I_J=\mathrm{diag}(1,1/2)$)

$$\mathrm{d}Y^I=\frac{1}{2}\mathrm{Tr}(\mathrm{d}\mathsf{U}\,\mathsf{U}^{-1}\mathsf{M}_I^\mathrm{T})\Rightarrow g_{IJ}\mathrm{d}Y^I\mathrm{d}Y^J=\frac{\mathrm{d}z^2+\frac{1}{2}\mathrm{d}y^2}{4z^2}\Rightarrow\mathbb{H}^2\text{ geometry!}$$

Hyperbolic geometry interpretation:
Poincaré half-plane and disk

$$(y_0, z_0) = (0, 1) \longrightarrow (y_1, z_1) = \left(-\frac{\varphi}{\sqrt{2}\mathcal{A}}, \frac{k}{\mathcal{A}}\right) \quad \xleftrightarrow{Z \equiv y + iz} \frac{Z - i}{Z + i}$$

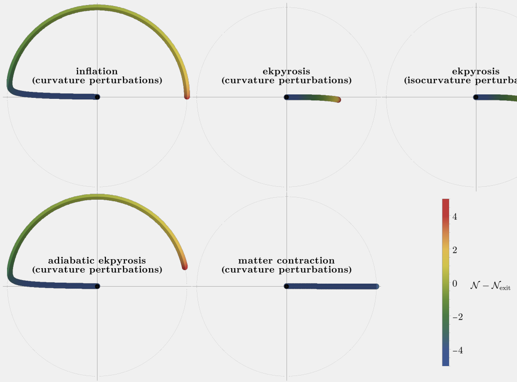


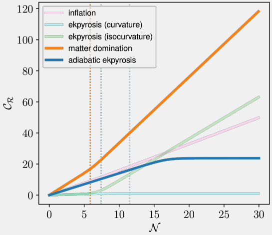
- Complexity = geodesic length:

$$\mathcal{C} = \frac{1}{2}\mathrm{argcosh}(X) = \frac{1}{2}\ln\left(X + \sqrt{X^2 - 1}\right) ,$$

$$X = \frac{z_0^2 + z_1^2 + \frac{1}{2}(y_1 - y_0)^2}{2 z_0 z_1} = \frac{1}{2}\left(\frac{\mathcal{A}}{k} + \frac{k}{\mathcal{A}} + \frac{\varphi^2}{k\mathcal{A}}\right)$$

Soon to be applied to early universe cosmology!





Cosmological perturbations — a quick review

- In GR, consider comoving curvature perturbations about FRW: $\delta g_{ij}(t,\mathbf{x}) = 2a(t)^2\mathcal{R}(t,\mathbf{x})\delta_{ij} \implies$

$$S^{(2)} = \int \mathrm{d}^3\mathbf{x}\mathrm{d}t\, a^3\epsilon \left(\dot{\mathcal{R}}^2 - \frac{(\partial_i\mathcal{R})^2}{a^2}\right), \qquad \epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3}{2}\left(1+\frac{p}{\rho}\right)$$

- The canonically normalised variable is $v \equiv z\mathcal{R}$, $z^2 \equiv 2\epsilon a^2$, such that (w/ $\mathrm{d}\tau \equiv a^{-1}\mathrm{d}t$ and going to Fourier space)

$$S^{(2)} = \frac{1}{2}\int \mathrm{d}^3\mathbf{k}\mathrm{d}\tau \left[(v_k')^2 - k^2 v_k^2 - 2\frac{z'}{z}v_k v_k' + \left(\frac{z'}{z}\right)^2 v_k^2\right]$$

$$\implies v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0 \qquad \text{(Mukhanov-Sasaki)}$$

- Quantize Fourier modes $v_k \to \hat{v}_{\mathbf{k}} = v_k(\tau)\hat{a}_{\mathbf{k}} + v_k^*(\tau)\hat{a}^\dagger_{-\mathbf{k}}$ with commutation relations $[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{-\mathbf{p}}] = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{p})$, so $\hat{a}_{\mathbf{k}}|0\rangle = 0$ and with conjugate momentum

$$\hat{\Pi} \equiv \frac{\partial S^{(2)}}{\partial v'} = \hat{v}' - \frac{z'}{z}\hat{v} \quad \Rightarrow \quad [\hat{v}(\tau,\mathbf{x}), \hat{\Pi}(\tau,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

The (less conventional) Schrödinger picture

- Quantum fluctuations are described by Gaussian wavefunctions (1-d harmonic oscill.):

$$|\Psi_k\rangle \propto \exp\left(-\frac{1}{2}A_{vv}v_k^2\right) = \exp\left(-\frac{1}{2}A_{\mathcal{R}\mathcal{R}}\mathcal{R}_k^2\right)$$

  A_{vv}, $A_{\mathcal{R}\mathcal{R}}$ = correlators

- Upon inverting, $i\hat{a}_{\mathbf{k}} = \left(v_k^{*\prime} - \frac{z'}{z}v_k^*\right)\hat{v}_{\mathbf{k}} - v_k^*\hat{\Pi}_{\mathbf{k}}$, so from the vacuum state $\hat{a}_{\mathbf{k}}|\Psi_k\rangle_{\mathrm{vac}} = 0 \Rightarrow$

$$A_{vv} = -i\frac{v_k^{*\prime}}{v_k^*} + i\frac{z'}{z}\,, \qquad A_{\mathcal{R}\mathcal{R}} = z^2 A_{vv}$$

- The Schrödinger eqn. $i|\Psi_k\rangle' = \hat{H}|\Psi_k\rangle \Leftrightarrow$ Heisenberg EOM (Mukhanov-Sasaki mode function)
- Bunch-Davies vacuum ($-k\tau \to \infty$):

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \Rightarrow \begin{cases} A_{vv} \simeq k \\ A_{\mathcal{R}\mathcal{R}} \simeq 2\epsilon k a(\tau)^2 \end{cases} \Rightarrow |\Psi_k\rangle_{\mathrm{vac}} = |\Psi_k\rangle_{\mathrm{R}} = \left(\frac{k}{\pi}\right)^{1/4} e^{-\frac{1}{2}kv_k^2}$$

Quantum-to-classical transition

- What is the large-scale/late-time ($-k\tau \to 0$) Target state?

$$|\Psi_k\rangle_{\mathrm{T}} = \left(\frac{\mathcal{A}}{\pi}\right)^{1/4} e^{-\frac{1}{2}(\mathcal{A}+i\varphi)v_k^2} \propto e^{-\frac{1}{2}A_{vv}v_k^2}$$

- We want fluctuations to classicalize. WKB $\Rightarrow$ phase of the wavefunction must vary much faster than its amplitude:

$$\left|\frac{\varphi}{\mathcal{A}}\right| = \left|\frac{\mathrm{Im}\,A_{vv}}{\mathrm{Re}\,A_{vv}}\right| \to \infty \quad \Rightarrow \quad \text{squeezed state}$$

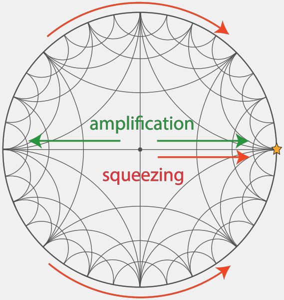
$$\mathcal{A} = \mathrm{Re}\,A_{vv} \to 0 \quad \Rightarrow \quad \text{amplified state}$$

$$(\Delta v)^2 \equiv \langle \hat{v}^2\rangle - \langle \hat{v}\rangle^2 = \langle \hat{v}^2\rangle = \frac{1}{2\mathcal{A}}\,, \qquad (\Delta\Pi)^2 = \frac{\mathcal{A}}{2}\left(1 + \frac{\varphi^2}{\mathcal{A}^2}\right)$$

Recall hyperbolic complexity

$$(y_0, z_0) = (0,1) \longrightarrow (y_1, z_1) = \left(-\frac{\varphi}{\sqrt{2}\mathcal{A}}, \frac{k}{\mathcal{A}}\right) \to (\pm\infty, \infty)$$

$$\overset{Z\equiv y+iz}{\longleftrightarrow} \frac{Z-i}{Z+i} \to 1+0i$$



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$$X = \frac{z_0^2 + z_1^2 + \frac{1}{2}(y_1 - y_0)^2}{2z_0 z_1} = \frac{k}{2\mathcal{A}} + \frac{\mathcal{A}}{2k}\left(1 + \frac{\varphi^2}{\mathcal{A}^2}\right) = k(\Delta v)^2 + \frac{(\Delta\Pi)^2}{k}$$

Let's apply it!

- Constant-EoS single-field models (adiabatic perturbations):

$$v_k'' + \left(k^2 - \frac{\alpha^2 - 1/4}{\tau^2}\right) v_k = 0\,, \qquad \alpha \equiv \frac{1}{2}\left|\frac{3-\epsilon}{1-\epsilon}\right|$$

$$\Rightarrow \quad v_k(\tau) = \frac{\sqrt{-\pi\tau}}{2} H_\alpha^{(1)}(-k\tau) \quad \Rightarrow \quad A_{vv} = ik\left(\frac{H_{\alpha-1}^{(1)}(-k\tau)}{H_\alpha^{(1)}(-k\tau)}\right)^*$$

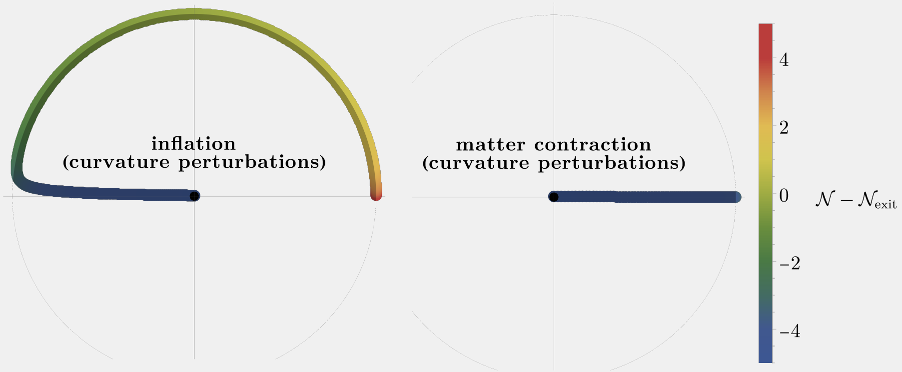
- Scale-invariant models ($\alpha \approx 3/2$) as $-k\tau \to 0$:
  - Inflation ($\epsilon \approx 0$):

$$A_{\mathcal{R}\mathcal{R}} \sim 1 + \frac{i}{\tau}\,, \quad \langle \hat{\mathcal{R}}^2 \rangle = \frac{1}{2\,\mathrm{Re}\,A_{\mathcal{R}\mathcal{R}}} \sim \mathrm{const.}\,, \quad \left|\frac{\mathrm{Im}\,A_{\mathcal{R}\mathcal{R}}}{\mathrm{Re}\,A_{\mathcal{R}\mathcal{R}}}\right| \sim \frac{1}{\tau} \to \infty$$

  - Matter contraction, a.k.a. matter bounce ($\epsilon \approx 3/2$):

$$A_{\mathcal{R}\mathcal{R}} \sim \tau^6 + i\tau^5\,, \quad \langle \hat{\mathcal{R}}^2 \rangle = \frac{1}{2\,\mathrm{Re}\,A_{\mathcal{R}\mathcal{R}}} \sim \frac{1}{\tau^6}\,, \quad \left|\frac{\mathrm{Im}\,A_{\mathcal{R}\mathcal{R}}}{\mathrm{Re}\,A_{\mathcal{R}\mathcal{R}}}\right| \sim \frac{1}{\tau}$$

Full evolution on the Poincaré disk



Lehners & JQ [2012.04911]

Ekpyrotic cosmology

- Scalar fields (σ, s) with steep negative exponential potentials, so with ultra-stiff EoS ($\epsilon > 3 \Leftrightarrow w > 1$), which drives slow contraction ($a \propto (-t)^{1/\epsilon}$)

$$V(\sigma, s) \simeq -V_0 e^{-\sqrt{2\epsilon}\sigma} \left[1 + \frac{\kappa_2}{2}\epsilon s^2\right], \quad \kappa_2 \sim \mathcal{O}(1)$$

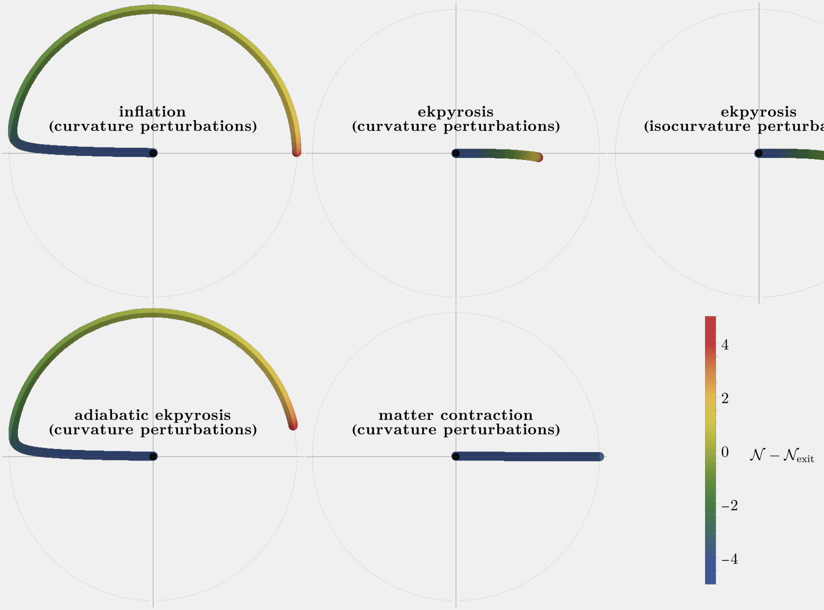
- Adiabatic (σ) perturbations aren't amplified and remain blue
- Entropy (or isocurvature; s) perturbations, $u \equiv a\,\delta s$:

$$u_k'' + \left(k^2 - \frac{a''}{a} + a^2\frac{\partial^2 V}{\partial s^2}\right)u_k = 0$$

$$\Rightarrow\ u_k(\tau) = \frac{\sqrt{-\pi\tau}}{2}H^{(1)}_{\alpha_s}(-k\tau), \quad \alpha_s \overset{\kappa_2\approx 1}{\approx} \sqrt{\frac{9}{4} - \frac{3}{\epsilon}} \overset{\epsilon\gg 3}{\approx} \frac{3}{2} \ \Rightarrow\ \text{scale inv.}$$

$$\Rightarrow\ A_{uu} = -\frac{i}{(\epsilon-1)\tau}\left[\left(\frac{1}{2} - \alpha_s\right)(\epsilon-1) - 1\right] + ik\frac{H^{(1)*}_{\alpha_s-1}(-k\tau)}{H^{(1)*}_{\alpha_s}(-k\tau)}$$

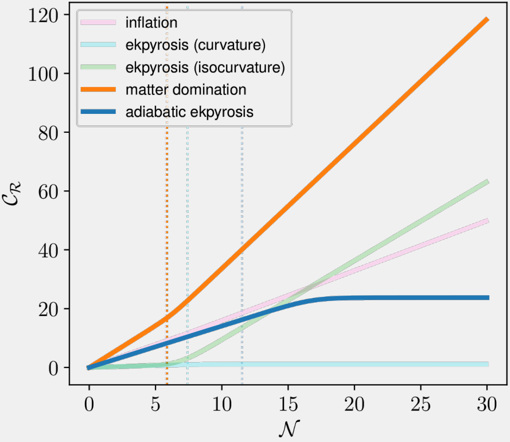
$$\Rightarrow\ A_{\delta s\delta s} \sim \tau^2 + \frac{i}{\tau}, \quad \langle\widehat{\delta s}^2\rangle = \frac{1}{2\,\mathrm{Re}\,A_{\delta s\delta s}} \sim \frac{1}{\tau^2}, \quad \left|\frac{\mathrm{Im}\,A_{\delta s\delta s}}{\mathrm{Re}\,A_{\delta s\delta s}}\right| \sim \frac{1}{\tau^3}$$



inflation
(curvature perturbations)
ekpyrosis
(curvature perturbations)
ekpyrosis
(isocurvature perturbations)
adiabatic ekpyrosis
(curvature perturbations)
matter contraction
(curvature perturbations)
4
2
0
−2
−4
$\mathcal{N} - \mathcal{N}_{\text{exit}}$

$$\text{super-horizon:} \qquad \Delta\mathcal{C}_{\mathcal{R}}^{\text{inf.}} \simeq \sqrt{2}(1+2\epsilon)\Delta\mathcal{N}\,,$$

$$\Delta\mathcal{C}_{\delta s}^{\text{ekp. (iso.)}} \simeq 2\sqrt{2}\left(\frac{\epsilon-\frac{3}{2}}{\epsilon-1}\right)\Delta\mathcal{N}\,, \qquad \Delta\mathcal{C}_{\mathcal{R}}^{\text{matter}} \simeq 3\sqrt{2}\Delta\mathcal{N}$$



Lehners & JQ [2012.04911]

Conclusions

- Complexity depends on very few things, mainly on the model in question and the number of e-folds $\longrightarrow$ useful **classifier** of models
- Very **mild dependence on specifics** within a given model, in particular mild dependence on EoS and no direct dependence on wavenumber nor on magnitude of the potential
- Even though all models must achieve precisely the same end state, both the representation on the Poincaré disk and the behaviour of complexity highlight the **differences** between models

Thank you for your attention!

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